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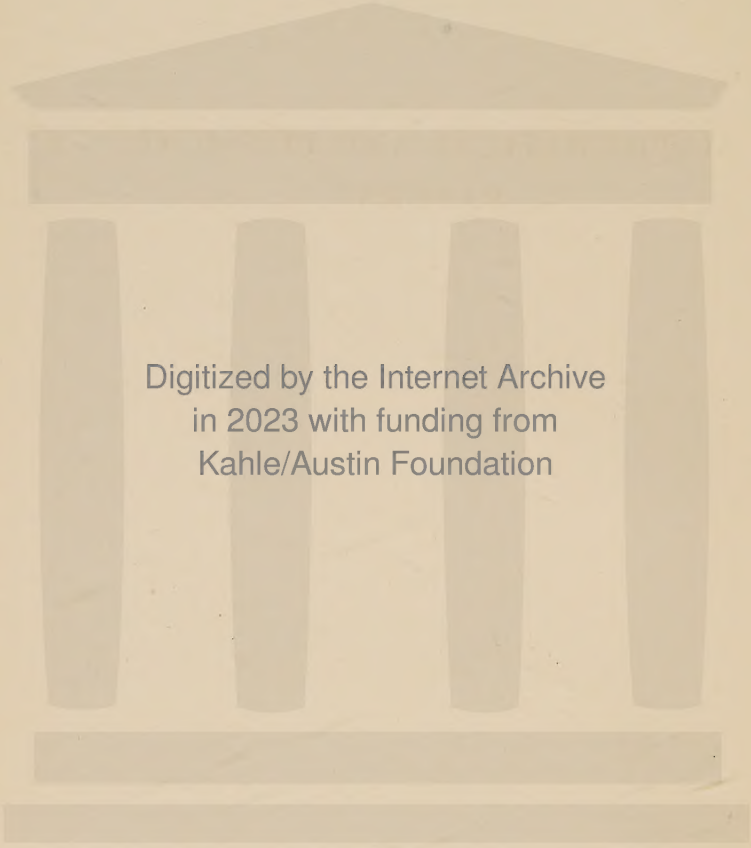
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MATHEMATICAL AND PHYSICAL PAPERS

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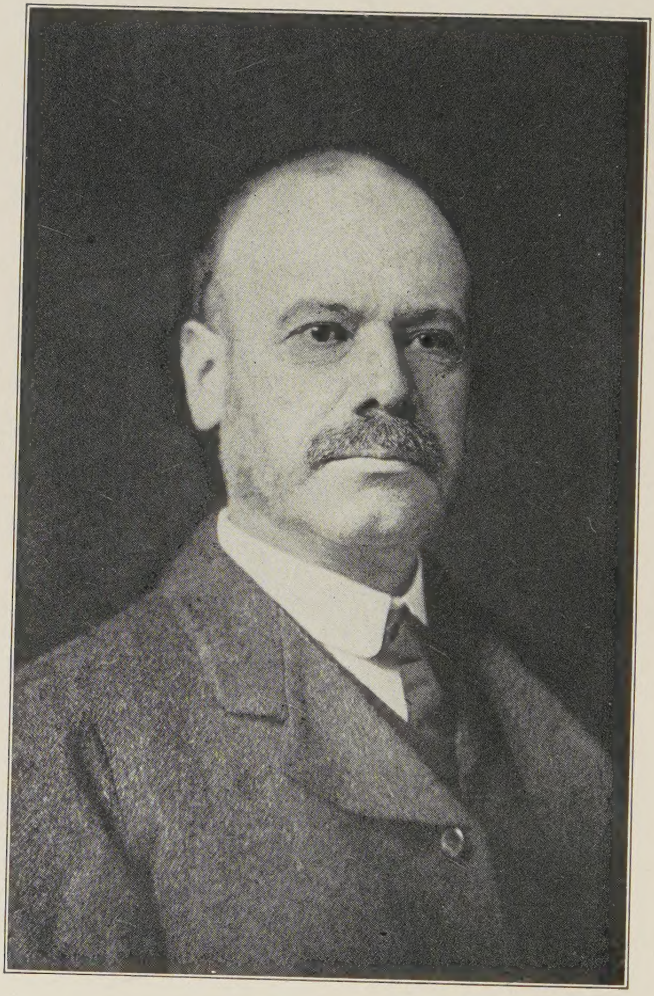

B. O. Peirce.

MATHEMATICAL AND PHYSICAL PAPERS

1903–1913

BY

BENJAMIN OSGOOD PEIRCE

LATE HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY IN HARVARD UNIVERSITY

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PREFACE

In this volume are assembled practically all the papers which Professor B. O. Peirce published in the last ten years of his life. The project originated with the late Professors John Trowbridge and Wallace C. Sabine, and has been further strengthened by the inquiries of scientists both here and abroad, to whom certain of the reprints were inaccessible.

Thanks are due to the American Academy of Arts and Sciences and to the *American Journal of Science* for kind permission to republish these papers.

Professors W. E. Byerly, E. H. Hall, W. F. Osgood, and Theodore Lyman have been most generous in advice and assistance.

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MATHEMATICAL AND PHYSICAL PAPERS

I

ON THE TEMPERATURE COEFFICIENTS OF MAGNETS MADE OF CHILLED CAST IRON[1]

Besides a number of d'Arsonval galvanometers, furnished with hardened forged-steel magnets, from the shops of well-known makers in America and in Europe, there are in the Physical Laboratories of Harvard University about thirty similar instruments in which the permanent fields are due to chilled and seasoned cast-iron magnets. These latter have proved very satisfactory, and, after a trial of three years, we are about to add to their number.

Although chilled cast-iron magnets were used years ago in some forms of telephones, straight magnets are most conveniently made of steel; indeed, if they are to be employed in measuring the intensity of the earth's field, the best tool steel, ground slowly into shape under water after the hardening, is not too homogeneous for the purpose. Steel for permanent magnets, however, needs special skill in the handling, if the results are to be satisfactory, and not every successful toolmaker knows how to forge and to harden, well and quickly, even a horseshoe magnet, unless it be of very simple form. Two straight, hollow bar magnets were made and ground most carefully for use in the Jefferson Physical Laboratory, by a firm which manufactures machine tools of the highest grade. These were supposed to be as nearly alike as possible, but they proved to be magnetically very different, for the permanent moment of one was twice that of the other.

Of late I have been using magnets made of soft iron castings, subsequently chilled, to furnish the artificial field in an oil-damped amperemeter, and in a similar voltmeter firmly set up in the laboratory; and, since it was desirable that the indications of these instruments should be trustworthy within one part in a thousand of their larger deflections, it became necessary to test the permanency of the magnets, and to determine their temperature coefficients. This paper gives the results of measurements made on a number of magnets of this kind.

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xxxviii, no. 19, February, 1903.

Most of the magnets experimented upon were made of castings chilled by Mr. G. W. Thompson, the mechanician of the Jefferson Physical Laboratory, who has had a great deal of experience with the process. They were first heated to a bright red in a gas furnace under a power blast, and then plunged into a cold acid bath kept in violent agitation. The castings thus hardened were relaxed by long exposure to boiling water or steam, then magnetized to saturation, and finally seasoned, after prolonged boiling, by being alternately heated in steam and cooled in tap water. The whole seasoning process reduced the magnetic moment of each specimen by perhaps twenty per cent of the value it had just after the magnetization. If, after a magnet has been seasoned, its temperature be suddenly raised from 0° C. to 100° C. and then as suddenly lowered again, it may not wholly recover its original strength until after the lapse of an hour or two; if, however, the range be only 40° or 50° C., I have been unable to detect any lag in the attainment of the whole of the original moment after the heating.

Although there is no advantage in using cast-iron for straight magnets, I had a number made for comparison with fine steel magnets of the same dimensions. The cast-iron magnets looked rough in comparison with the others, but the moments of a large number of them seemed to differ less among themselves than the moments of the same number of the steel magnets. The strongest steel magnet that I tested had a moment about four per cent greater than that of the strongest cast-iron magnet, but the average moment of the cast-iron magnets was practically the same as (in fact two per cent greater than) the average of the seasoned steel magnets.

In determining the temperature coefficients, the straight magnet to be experimented upon was fixed firmly in a non-magnetic holder inside a non-magnetic tube, so as to be in Gauss's A Position east of a mirror magnetometer. By the help of a system of pipes and cocks, tap water, steam, or a stream from a bath-water heater at almost any desired temperature, between 15° C. and 100° C., could be sent through the tube containing the magnet. On the west of the magnetometer, so placed in Gauss's A Position as to bring the needle back exactly into the meridian, was a short, seasoned, compensating magnet, fixed wholly within a wooden holder and completely shielded from sudden temperature changes. If α_0 is the needle deflection which the com-

pensating magnet would cause if the magnet to be tested were removed, M_0, the moment of the last-named magnet at the temperature t_0 at which the adjustments have been made, and M, the moment of this magnet when, its temperature having become raised to t, the needle is deflected through the angle α,

$$\frac{M_0 - M}{M_0} = \frac{\tan \alpha}{\tan \alpha_0}.$$

Since the temperature coefficients of seasoned bar magnets of a given length and of given material are, in general, larger the greater the cross section of the bar, it is necessary in comparing materials to take magnets of nearly the same dimensions. Besides a number of chilled cast-iron magnets 18 centimeters long and about 0.95 centimeters in diameter, I had many carefully made steel magnets of the same area of cross section and of almost the same length. In the case of all these, the rate of loss of moment per degree of rise of temperature was greater at higher temperatures than at low; we may, however, for the purpose of comparison, use the mean loss, per degree, of the magnetic moment, when the magnet is heated from about 10° C. to 100° C., expressed in terms of the moment at the lower temperature. These mean losses were found to be

0.00042 in the case of the seasoned chilled iron magnets;
0.00046 in the case of the seasoned magnets made of "Crescent Steel Drill Rod";
0.00046 in the case of the seasoned magnets made of Jessop's Round Black Tool Steel;
0.00070 in the case of the seasoned magnets made of Jessop's Square Tool Steel.

I had bar magnets made of many other materials, for instance, of Jessop's and Mushet's self-hardening steels, but none of exactly the dimensions of the cast-iron magnets. No kind of steel that I tested had, however, when proper allowance was made for dimensions, quite so small a temperature coefficient as the chilled iron.

The mean temperature coefficient of chilled cast-iron magnets 18 centimeters long and 1.25 centimeters in diameter, as obtained from a number of specimens, was 0.00056, which is very low.

The forms of some of the magnets which we have used (either singly or with others of the same kind) in various instruments are shown in

the subjoined figure. The shapes marked 1, 2, 3, 6, are employed, with the long way of the opening between the poles vertical, in d'Arsonval galvanometers; two or three castings of the shape marked 4, and a number of thin plates of the shape marked 8, are used together in other

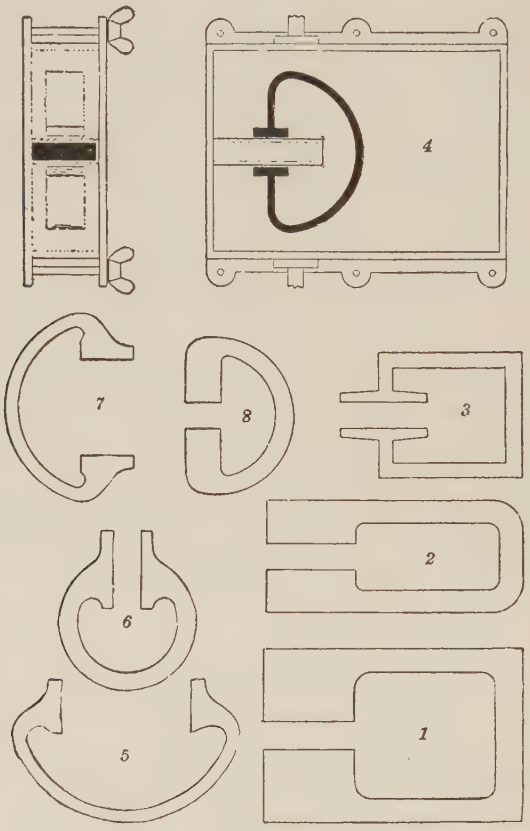


FIGURE 1.

instruments of the same kind. Magnets of the shapes marked 5 and 7 produce the artificial fields in laboratory amperemeters and voltmeters.

For our present purpose we may define the temperature coefficients of one of these magnets as the rate of change of the whole magnetic induction across a given surface between the poles, when the temperature of the magnet is raised by one degree. This can be measured with sufficient accuracy by pulling out, from a definite position between the

jaws, a coil of suitable shape made of manganine wire and connected with a ballistic galvanometer. In order to be able to make the determinations conveniently, I had a brass box made, of the shape indicated in the figure. The box itself was first cast in one piece, and then a slot for the coil was cut on a milling machine, and a rectangular cavity, open to the outside air but closed to the inside of the box, was constructed by soldering two thin pieces of brass into the end and top of the slot. Into this cavity a set of forms carrying thin coils of the shapes needed, fitted exactly. The box itself, and the cover, were mounted on the face-plate of a lathe and turned off smooth, so that when a piece of rubber packing was inserted between the two, and the whole was screwed together, the case thus made was water-tight. The box was mounted on a wooden frame which had sliders for the forms which carried the coils. The magnet to be tested was fastened firmly in place by a holder not shown in the figure, and the box was connected with a set of pipes, so that cold water, warm water, or steam could be sent through it at pleasure.

The temperature coefficient of a bent cast-iron magnet, as defined above, generally increases with the temperature, but for purposes of comparison, we may use the mean value K of this coefficient between 10° C. and 100° C.

Three magnets of the form marked 1, chilled by Mr. Thompson and weighing as much as 1250 grams each (nearly three pounds), gave for K the values 0.00036, 0.00037, and 0.00034 respectively; another magnet of the same pattern, treated by a maker of hardened cast-iron machinery, yielded the value 0.00082. Whatever the secret process employed in this last case, the resulting magnet was by no means so useful as those made from castings chilled in the manner described above.

Unchilled castings make very undesirable magnets, for the temperature coefficients are usually five or six times as large as in the case of chilled magnets, and it seems impossible to get their magnetic moments really permanent. Curiously enough, the chilling process makes a casting less brittle than before, and causes the grain of a fracture to be finer and more uniform.

The values of K seem to indicate that the whole interior of the casting is affected by the chilling, whereas it is extremely difficult to

harden a thick piece of steel uniformly. It did not appear that a magnet made up of a lot of thin plates chilled separately had a smaller temperature coefficient than a solid magnet of the same dimensions.

Castings of the shapes marked 3, 4, and 6 weighed about 260 grams, 160 grams, and 500 grams, respectively, and yielded for K the values 0.00040, 0.00040, 0.00031. The actual temperature coefficients at low temperatures are always less than these mean values, and in the case of the last-mentioned form the coefficient is not greater than 0.00013 between 10° C. and 40° C. I have myself never found a value quite so small as this for a massive steel magnet, though several observers have obtained extremely low coefficients for very slender steel wires, and even negative coefficients for comparatively weak magnets made of some alloys.

Using such chilled magnets as I have described, and employing composite galvanometer coils of manganine and copper, with permanent manganine shunts, it is not difficult to make a cheap fixed amperemeter, the indications of which shall be almost wholly independent of the room temperature. In the case of a d'Arsonval galvanometer of the usual form, slight temperature changes in the torsional rigidity of the suspension wire have to be taken into account.

II

ON FAMILIES OF CURVES WHICH ARE THE LINES OF CERTAIN PLANE VECTORS EITHER SOLENOIDAL OR LAMELLAR [1]

If a vector function has no component parallel to the axis of z, and if the tensors of its components taken parallel to the axes of x and y can be expressed by the scalar point functions $X = \phi_1(x, y)$, $Y = \phi_2(x, y)$, which are independent of z, every line of the vector is a curve parallel to the xy plane, defined by the equations $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{0}$, and it is sometimes convenient to call the vector itself "plane," and to say that it is "coplanar with" $z = 0$. The projection on the xy plane of any line of such a vector is itself a line of the vector, and a survey of the whole field can be obtained by studying the lines which lie in this plane.

The "divergence" of a vector coplanar with the xy plane is the quantity $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$, and the "curl" of the vector is a vector, directed parallel to the z axis, of intensity $\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$. If the divergence is zero in any region, the vector is said to be "solenoidal" in that region; a vector the curl of which vanishes is said to be "lamellar."

Given any family of curves in the xy plane represented by the equation $u \equiv f_1(x, y) = c_1$, it is possible to find an infinite number of plane vectors which have the u curves as lines, by assuming in each case X at pleasure, and then making

$$Y = -X \cdot \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

The vector (X_0, Y_0) and the vector $(R \cdot X_0, R \cdot Y_0)$, where R is any function of xy, evidently have the same lines, and, if (X_0, Y_0) has for lines the u curves, no other vector has the same lines unless it is of the

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xxxviii, no. 24, May, 1903.

form $(R \cdot X_0, R \cdot Y_0)$. Of all the vectors which have the u curves for lines some are lamellar, for, if v is any function orthogonal to u, defined by the equation

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0,$$

so that the curves of the families $u = c_1, v = c_2$ cut one another at right angles, the vector which has the components $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$ has for its lines the u curves, and it is lamellar, since

$$\frac{\partial^2 v}{\partial x \cdot \partial y} = \frac{\partial^2 v}{\partial y \cdot \partial x}.$$

If (X_0, Y_0) which has the u curves for lines is lamellar, so is the vector $[X_0 \cdot F(v), Y_0 \cdot F(v)]$, where F represents any ordinary function; and no lamellar vector has the same lines unless it is of the form just given.

If (X_1, Y_1) is a solenoidal vector which has the u curves for its lines, the vector $[X_1 \cdot F(u), Y_1 \cdot F(u)]$ has the same lines and is also solenoidal; no solenoidal vector has these lines unless it can be written in this form. It will soon appear that of all the vectors the lines of which are the u curves, some are always solenoidal, but no vector which has these curves for lines can be both solenoidal and lamellar, unless u happens to satisfy Lamé's condition for isothermal parameters,[1] that is, unless $\frac{\nabla^2(u)}{h_u{}^2}$ is expressible as a function of u alone, where

$$\nabla^2(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad h_u{}^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

If a set of orthogonal curvilinear coördinates in the xy plane be defined by the functions

$$u = f_1(x, y), \quad v = f_2(x, y);$$

and if

$$U = \xi(x, y), \quad V = \eta(x, y)$$

represent the magnitudes, at the point (x, y), of the components, taken in the directions in which u and v increase most rapidly, of a vector, Q, coplanar with $z = 0$; it is not difficult to prove, by direct transforma-

[1] Lamé, *Leçons sur les coordonnées curvilignes*, p. 31; *Leçons sur les fonctions inverses*, p. 5; Somoff-Ziwet, *Theoretische Mechanik*, i, 113 and 128.

tion or otherwise, that the divergence of Q is given by the well-known expression

$$\text{Div. } Q \equiv \left(\frac{U}{h_u}\right)\nabla^2(u) + h_u^2 \cdot \frac{\partial}{\partial u}\left(\frac{U}{h_u}\right) + \left(\frac{V}{h_v}\right)\nabla^2(v) + h_v^2 \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_v}\right) \quad (1)$$

and that

$$\text{Tensor curl } Q \equiv h_u \cdot h_v \left[\frac{\partial}{\partial u}\left(\frac{V}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{U}{h_u}\right)\right]. \quad (2)$$

If the lines of Q coincide with the u curves, the vector has no component perpendicular to these curves and U is everywhere equal to zero, so that

$$\text{Div. } Q \equiv \frac{V}{h_v} \cdot \nabla^2(v) + h_v^2 \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_v}\right), \quad (3)$$

$$\text{Tensor curl } Q \equiv h_u \cdot h_v \cdot \frac{\partial}{\partial u}\left(\frac{V}{h_v}\right), \quad (4)$$

where h_v is the gradient of v.

In applying these expressions it is convenient to remember that

$$\frac{\nabla^2(u)}{h_u^2} = \frac{\partial \log\left(\frac{h_u}{h_v}\right)}{\partial u}, \qquad \frac{\nabla^2(v)}{h_v^2} = \frac{\partial \log\left(\frac{h_v}{h_u}\right)}{\partial v}.$$

It is easy to see from (3) and (4) that the statements which follow are true:

(a) If V is to be solenoidal,[1] we must have

$$\frac{\partial}{\partial v}\left(\log \frac{V}{h_v}\right) = -\frac{\nabla^2(v)}{h_v^2}. \quad (5)$$

The second member of this equation is expressible as a function of u and v; if it be integrated with respect to v while u is considered constant, and if the arbitrary function $\chi(u)$ be added to the result, we shall get $\psi(u, v) + \chi(u)$ the partial derivative of which with respect to v is

$$-\frac{\nabla^2(v)}{h_v^2}; \text{ then } V \equiv h_v \cdot e^{\chi(u)} \cdot e^{\psi(u,v)}. \quad (6)$$

(b) If V is to be lamellar, we may write

$$V \equiv h_v \cdot \tau(v). \quad (7)$$

[1] See equation (18).

where τ is any ordinary function. Its divergence is

$$\tau(v) \cdot \nabla^2(v) + h_v^2 \cdot \tau'(v).$$

If V is to be solenoidal as well as lamellar, we may obtain Lamé's condition immediately by substituting the value of V from (7) in (5).

(*c*) If, like the vector which defines the field of electromagnetic force within an infinitely long cylinder of revolution which carries lengthwise a uniformly distributed, steady current of electricity, V is solenoidal and a function of u only, we must have

$$V\left[\frac{\nabla^2(v)}{h_v} + h_v^2 \frac{\partial}{\partial v}\left(\frac{1}{h_v}\right)\right] = 0,$$

or

$$2 \cdot \nabla^2(v) = -\frac{\partial(h_v^2)}{\partial v}. \tag{8}$$

(*d*) If, like the attraction within a homogeneous, infinitely long, cylinder of revolution, V is lamellar and a function of v only, the gradient of v cannot involve u, so that

$$h_v = f(v), \text{ where } f \text{ is arbitrary, or } \frac{\partial h_v}{\partial u} = 0. \tag{9}$$

(*e*) If V is lamellar and a function of u only, $\frac{V}{h_v}$ must be independent of u and

$$\frac{\frac{\partial h_v}{\partial u}}{h_v} \text{ is a function of } u \text{ only.} \tag{10}$$

In this case h_v is either a function of u only or is expressible as the product of a function of u and a function of v.

(*f*) If V is to be solenoidal [1] and a function of v only, the expression

$$\frac{\frac{\partial h_v}{\partial v}}{h_v} - \frac{\nabla^2(v)}{h_v^2} = \frac{\frac{dV}{dv}}{V} \tag{11}$$

must be either constant or expressible in terms of v. If V is not lamellar, h_v must in this case involve u.

(*g*) If V is lamellar and if Ω is a scalar potential function of V, Ω must be expressible in terms of v and the divergence of V is equal to

$$\frac{d\Omega}{dv} \cdot \nabla^2(v) + \frac{d^2\Omega}{dv^2} \cdot h_v^2. \tag{12}$$

[1] See equation (27).

(*h*) If the tensor of V has the same value for all values of x and y, V is lamellar if, and only if, h_v is constant or expressible in terms of v; it is solenoidal if, and only if,

$$2\nabla^2(v) = \frac{\partial h_v^2}{\partial v}. \tag{8}$$

(*i*) Whatever u is, the vector which has the components

$$X_1 = \frac{f(u)}{h_v} \cdot \frac{\partial v}{\partial x}, \qquad Y_1 = \frac{f(u)}{h_v} \cdot \frac{\partial v}{\partial y},$$

and the vector which has the components

$$X_2 = \frac{\phi(v)}{h_v} \cdot \frac{\partial v}{\partial x}, \qquad Y_2 = \frac{\phi(v)}{h_v} \cdot \frac{\partial v}{\partial y},$$

have the u curves for lines. The tensor of the first is a function of u only, that of the second a function of v only.

(*j*) If a solenoidal vector has the u lines for curves, its curl must be of the form $\phi(u) \cdot \nabla^2(u) + \phi'(u) \cdot h_u^2$, where ϕ is arbitrary. If, for instance, the u curves are concentric circumferences, the curl of the vector must be expressible as a function of the distance from the centre.

(*k*) If the tensor of a vector V which has the u curves for lines is a function of u only, its divergence is of the form $V\left(\frac{\nabla^2(v)}{h_v} - \frac{\partial h_v}{\partial v}\right)$. If the u curves are concentric circumferences, V must be solenoidal.

(*l*) If the tensor of V is expressible in terms of v, the tensor of its curl is $-V\frac{h_u}{h_v} \cdot \frac{\partial h_v}{\partial u}$. If the u curves are straight lines emanating from a point, the curl is zero and the divergence a function of the distance from the point. The velocity in the case of a steady squirt [1] motion of a gas illustrates this.

The Gradients of Functions of Two Independent Variables

Before we consider briefly some of the equations of condition which have just been stated, it will be well to make a few simple statements concerning the gradients [2] of functions of x and y.

[1] Minchin, *Uniplanar Kinematics*, p. 178, examples 21 and 22.

[2] Lamé, *Leçons sur les coordonnées curvilignes*, p. 6; Maxwell, *Treatise on Electricity and Magnetism*, § 17.

The gradient of a function may or may not be expressible in terms of the function itself. The gradients of the expressions $(x^2 + y^2)$. $(x^2 - y^2)$ illustrate these two cases.

If the gradient of a function v is equal to $f(v)$, it is possible to form a function of v, $a\int\frac{dv}{f(v)}$, the gradient of which is constant.

If the gradient of a function v is equal to the constant a, it is possible to form two functions of v, namely $\frac{bv}{a}$ and $\frac{1}{a}\int f(v)\,dv$, the gradients of which are equal, respectively, to the arbitrarily chosen constant b and to the arbitrary function $f(v)$.

If the gradient of a function v is either constant or expressible in terms of v, the gradient of any differentiable function of v is expressible as a function of v.

If h_v is neither constant nor expressible in terms of v, no function of v exists the gradient of which is expressible in terms of v.

Since the gradients of two conjugate functions are numerically equal, it is clear that if h_v is expressible in terms of v, not all other functions the gradients of which are functions of v, are themselves expressible in terms of v.

If, for x and y in the expression

$$h_v^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2,$$

the quantities $\lambda = G(x, y)$, $\mu = H(x, y)$ be substituted, we shall obtain the new expression

$$h_v^2 = h_\lambda^2\left(\frac{\partial u}{\partial \lambda}\right)^2 + h_\mu^2\left(\frac{\partial u}{\partial \mu}\right)^2 + 2\left(\frac{\partial \lambda}{\partial x}\cdot\frac{\partial \mu}{\partial x} + \frac{\partial \lambda}{\partial y}\cdot\frac{\partial \mu}{\partial y}\right)\left(\frac{\partial v}{\partial \lambda}\cdot\frac{\partial v}{\partial \mu}\right) \quad (13)$$

and if we write $\lambda = (x + yi)$, $\mu = (x - yi)$,

$$h_v^2 = 4\,\frac{\partial v}{\partial \lambda}\cdot\frac{\partial v}{\partial \mu}\,: \quad (14)$$

from this last equation it is evident that if the gradient of v vanishes, v is either a function of $x + yi$ or a function of $x - yi$.

It is often convenient in dealing with differential equations which involve the gradients of functions, to use the independent variables of

equation (14) and we may note that u and v, two functions of λ and μ, are conjugate if, and only if,

$$\frac{\partial u}{\partial \lambda} = i\frac{\partial v}{\partial \lambda}, \qquad \frac{\partial u}{\partial \mu} = -i\frac{\partial v}{\partial \mu}. \tag{15}$$

If u and v are orthogonal functions,

$$\frac{\partial u}{\partial \lambda}\cdot\frac{\partial v}{\partial \mu} + \frac{\partial u}{\partial \mu}\cdot\frac{\partial v}{\partial \lambda} = 0. \tag{16}$$

If the gradients of u and v, two real functions of x and y, are everywhere equal while the directions of their gradient vectors are different,

$$\frac{\partial(u-v)}{\partial x}\cdot\frac{\partial(u+v)}{\partial x} + \frac{\partial(u-v)}{\partial y}\cdot\frac{\partial(u+v)}{\partial y} = 0 \tag{17}$$

and the functions $(u - v)$ and $(u + v)$ are orthogonal. The converse of this statement is true. If two orthogonal functions have equal gradients these functions are conjugate.

If the gradient vectors of two functions have the same direction at every point of the xy plane, one of these functions is expressible in terms of the other.

The quantities $u = \cos(bx - y)$, $v = \sin(by + x)$ illustrate the fact that the gradient of each of two orthogonal functions may be expressible in terms of the function itself.

The quantities $u = x^2 + y^2$, $v = \tan^{-1}\left(\frac{y}{x}\right)$ illustrate the fact that gradients of both of two orthogonal functions may be expressible in terms of one of the functions.

If the gradient of v, one of two orthogonal functions (u, v), is expressible in terms of u, or is constant, no other but a linear function of v has a gradient expressible in terms of u.

If the gradient of each of two orthogonal functions (u, v) is expressible as a product of a function of u and a function of v, so that

$$h_u = f(u)\cdot F(v), \qquad h_v = \phi(u), \qquad \psi(v),$$

it is possible to find two functions, $\int\frac{du}{f(u)}$, $\int\frac{dv}{\psi(v)}$, of u and v respectively, the gradient of each of which is expressible in terms of the other.

A solution of Laplace's Equation and any function of its conjugate

are orthogonal functions the ratio of the gradients of which is a function of the second function.

Vector Potential Functions of Plane Solenoidal Vectors

If u, v define a system of orthogonal curvilinear coördinates in the xy plane, and if Q_u, Q_v, Q_z are the components of a vector Q, taken in the directions in which u, v, z increase most rapidly, the components of the curl of Q in these directions are

$$h_v\left[\frac{\partial Q_z}{\partial v} - \frac{\partial}{\partial z}\left(\frac{Q_v}{h_v}\right)\right], \qquad h_u\left[\frac{\partial}{\partial z}\left(\frac{Q_u}{h_u}\right) - \frac{\partial Q_z}{\partial u}\right],$$

$$h_u h_v\left[\frac{\partial}{\partial u}\left(\frac{Q_v}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{Q_u}{h_u}\right)\right].$$

We may denote these quantities by K_u, K_v, K_z, respectively.

If Q is to be a vector potential function of a given solenoidal plane vector $(0, V, 0)$, which has the u curves for lines, we may assume that the components of Q involve u and v only, and since, in this case, $K_u = 0$, $K_v = V$, write $Q_z = F(u)$, where $V = -h_u \cdot \frac{dF(u)}{du}$. Any vector of the form $[Q_u, Q_v, F(u)]$, where Q_u, Q_v, are any functions of u and v subject only to the condition $\frac{\partial}{\partial v}\left(\frac{Q_u}{h_u}\right) = \frac{\partial}{\partial u}\left(\frac{Q_v}{h_v}\right)$, is a vector potential function of a solenoidal vector which has the u curves as lines, and there is no vector of this latter kind which does not have as a vector potential a vector of the form just given. In most cases it is simplest to make $Q_u = Q_v = 0$.

If, now, we ask what condition must be satisfied by the function u in order that the curves of the family $u = c$ may be the lines of a vector the tensor of which involves u only, we learn that, since V is of the form $-h_u \cdot F'(u)$, it is necessary and sufficient that h_u be a function of u only. That is

$$\frac{\partial h_u}{\partial v} = 0. \tag{18}$$

Since the divergence of any vector V which has the u curves for lines may be written in the form $h_u \cdot h_v \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_u}\right)$ as well as in the form (3), the condition stated in equation (18) is at once obtained.

If we denote the quantities

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \cdot \partial y}, \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \cdot \partial y}, \frac{\partial^2 v}{\partial y^2},$$

by $p, q, r, s, t, p', q', r', s', t'$, respectively, we have, since u and v are orthogonal,

$$pp' + qq' = 0; \tag{19}$$

whence by differentiation we get

$$p'r + pr' + q's + qs' = 0, \tag{20}$$

$$p's + ps' + q't + qt' = 0. \tag{21}$$

We have, moreover, $\frac{\partial x}{\partial u} = \frac{h_x}{h_u} \cdot \cos(x, u) = \frac{1}{h_u} \cdot \frac{p}{h_u} = \frac{p}{h_u^2}$,

and, similarly, $\frac{\partial y}{\partial u} = \frac{q}{h_u^2}, \quad \frac{\partial x}{\partial v} = \frac{p'}{h_v^2}, \quad \frac{\partial y}{\partial v} = \frac{q'}{h_v^2}, \tag{22}$

so that $$\frac{\partial h_u}{\partial u} = \frac{\partial h_u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial h_u}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{1}{h_u^2}\left(\frac{\partial h_u}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial h_u}{\partial y} \cdot \frac{\partial u}{\partial y}\right)$$

$$= \frac{p^2 r + 2pqs + q^2 t}{h_u^3}, \tag{23}$$

and $$\frac{\partial h_u}{\partial v} = \frac{\partial h_u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial h_u}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{1}{h_v^2}\left(\frac{\partial h_u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial h_u}{\partial y} \cdot \frac{dv}{\partial y}\right)$$

$$= \frac{p'[pq(r - t) + s(q^2 - p^2)]}{qh_u h_v^2}.$$

Since, however, $h_u^2 \equiv p^2 + q^2$, $h_v^2 \equiv p'^2 + q'^2$; $q^2 \cdot h_v^2 = p'^2 \cdot h_u^2$,

and $$\frac{\partial h_u}{\partial v} = \pm \frac{(pqr + (q^2 - p^2)s - pqt)}{h_v \cdot h_u^2}. \tag{24}$$

Equation (18) is equivalent, therefore, to the equation

$$pqr + (q^2 - p^2)s - pqt = 0. \tag{25}$$

If for r and t in (25) we substitute their values as obtained from (20) and (21), we shall get the equation

$$q'^2 r' - 2p'q's' + p'^2 t' = 0, \tag{26}$$

and this is (8) in expanded form.

If equation (18) or its equivalent (26) is satisfied, it is evident that by choosing $F(u)$ at pleasure we may find an infinite number of sole-

noidal vectors which have the u curves as lines and have tensors which involve u only.

A comparison of equations (9) and (18) shows that the condition that the u curves be possible lines of a set of solenoidal vectors the tensors of which involve u only, is the condition that the v curves be possible lines of a set of lamellar vectors the tensors of which involve u only.

If Q is a vector potential function of a solenoidal vector which has the u curves for lines, and a tensor expressible in terms of v, $-h_u F'(u)$ is a function of v, and h_u must be expressible as the product of a function of u and a function of v, that is,

$$h_u = f(u) \cdot \phi(v). \tag{27}$$

If for u in this differential equation we substitute w, defined by the equation $w \equiv \int \frac{du}{f(u)}$, we get the simpler equation

$$h_w = \phi(v) \text{ or } \frac{\partial h_w}{\partial w} = 0. \tag{28}$$

It is to be noticed that w has the same lines as u, and that (27) and (28) define the same curves; the equations (11) and (28) are evidently equivalent.

If u is such that a solenoidal vector, V, can be found which has the u curves for lines and a tensor expressible in terms of v, its x and y components are $\left(\frac{V}{h_v} \cdot \frac{\partial v}{\partial x}, \frac{V}{h_v} \cdot \frac{\partial v}{\partial y}\right)$. If we denote these components by X, Y, every other solenoidal vector which has the same lines has components of the form $X \cdot \psi(u)$, $Y \cdot \psi(u)$, and the vector is not a function of v alone unless the factor $\psi(u)$ degenerates into a constant, and the vector is a simple multiple of V.

A comparison of (10) and (27) shows that if the u curves are possible lines of a solenoidal vector the tensor of which is expressible in terms of v, the v curves are the possible lines of a lamellar vector the tensor of which is a function of v only.

If $\frac{\partial h_u}{\partial v} = 0$ and $\frac{\partial h_v}{\partial u} = 0$, the u curves are the lines of a set of solenoidal vectors the curls of which are expressible in terms of u only; and

the v curves are the lines of a set of solenoidal vectors the curls of which are expressible in terms of v only.

Possible Systems of Isothermal Straight Lines and Isothermal Circles in a Plane

(1) Let $\alpha x + \beta y = 1$, where α and β are any functions of a single parameter u, represent a family of straight lines in the xy plane, then we may write

$$\frac{\partial u}{\partial x} = \frac{-\alpha}{\alpha' x + \beta' y}, \quad \frac{\partial u}{\partial y} = \frac{-\beta}{\alpha' x + \beta' y}, \quad h_u{}^2 = \frac{\alpha^2 + \beta^2}{(\alpha' x + \beta' y)^2},$$

$$\frac{\nabla^2(u)}{h_u{}^2} = \frac{2(\alpha\alpha' + \beta\beta')}{\alpha^2 + \beta^2} - \frac{\alpha'' x + \beta'' y}{\alpha' x + \beta' y}. \tag{29}$$

If then $\frac{\nabla^2(u)}{h_u{}^2}$ is to be a function of u only, the last term in the second member of this last equation must be expressible in terms of u only, and we have $\alpha' = 0$, or $\beta' = 0$, or, in general, $\alpha'' : \alpha' = \beta'' : \beta'$, so that $\alpha = c\beta + d$, where c and d are constants of integration. The equation of the family of lines must be of the form $(c\beta + d)\, x + \beta y = 1$, and the lines all pass through the fixed point $\left(\frac{1}{d}, -\frac{c}{d}\right)$, which may be chosen at pleasure. If $d = 0$, the lines are parallel.

(2) Let $x^2 + y^2 - 2\alpha x - 2\beta y = \gamma$, where α, β, γ are functions of a single parameter u, represent a family of circumferences in the xy plane, then we may write

$$\frac{\partial u}{\partial x} = \frac{2(x - \alpha)}{2\alpha' x + 2\beta' y + \gamma'}, \qquad \frac{\partial u}{\partial y} = \frac{2(y - \beta)}{2\alpha' x + 2\beta' y + \gamma'},$$

$$h_u{}^2 = \frac{4(\alpha^2 + \beta^2 + \gamma)}{(2\alpha' x + 2\beta' y + \gamma')^2},$$

$$\frac{\nabla^2(u)}{h_u{}^2} = \frac{2\alpha\alpha' + 2\beta\beta' + \gamma'}{\alpha^2 + \beta^2 + \gamma} - \frac{1}{\alpha^2 + \beta^2 + \gamma} \cdot \frac{2\alpha'' x + 2\beta'' y + \gamma''}{2\alpha' x + 2\beta' y + \gamma'}. \tag{30}$$

If $\frac{\nabla^2(u)}{h_u{}^2}$ is a function of u only, the last term in the second member of (30) must be expressible in terms of u only. If $\alpha' = \beta' = 0$, we have a family of concentric circumferences. In general we may write $\alpha'' : \alpha' = \beta'' : \beta' = \gamma'' : \gamma'$, or $\beta = m\alpha + n$, $\gamma = 2k\alpha + l$, so that the

equation of the circles must be of the form

$$x^2 + y^2 - 2\,\alpha x - 2\,y\,(m\alpha + n) - 2\,k\alpha - l = 0, \tag{31}$$

where α is the only parameter. If we represent the first member of this equation by S_α, the equation $S_{\alpha_1} - S_{\alpha_2} = 0$ represents the straight line through the points of intersection of the circles which correspond to the two values α_1, α_2 of the parameter. In this case the line is $x + my + k = 0$, whatever the values of α_1 and α_2, therefore, as is well known, the system of isothermal circles [1] must pass through two fixed, real or imaginary, points.

Functions the Gradients of which are expressible in Terms of the Functions themselves

Several of the conditions stated in the previous pages [see (d), (h), and equation (18)] require that the gradient of a function be expressible in terms of the function itself, so that the normal derivative of the function has the same numerical value at all points of any one of its curves of level. We may state this requirement in a somewhat simpler form, however, if we remember that since the gradient of any function, ϕ, of u is equal to $\phi'(u) \cdot h_u$, the lines of all functions which satisfy the equation $h_u = f(u)$, whatever f may be, are included in the lines of functions which satisfy the equation $h_u = k$, where k is any constant (for instance 1). Every such family of lines forms a set of parallel curves. We have to solve, then, the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = k^2, \tag{32}$$

one of the standard forms for partial differential equations of the first order.

Its complete integral is

$$u = ax + y\sqrt{k^2 - a^2} + c,$$

and its general integral,

$$u = ax + y\sqrt{k^2 - a^2} + \psi(a), \tag{33}$$

subject to the condition

$$0 = x - \frac{ay}{\sqrt{k^2 - a^2}} + \psi'(a).$$

[1] Darboux, *Leçons sur la théorie générale des surfaces.*

The equation $h_u = k$ is also equivalent to the equation,

$$\frac{\partial u}{\partial \lambda} \cdot \frac{\partial u}{\partial \mu} = 1 \tag{34}$$

The complete integral of (35) is

$$u = a\lambda + \frac{\mu}{a} + c,$$

and its general integral [1] may be found by eliminating a between the equations,

$$u = a\lambda + \frac{\mu}{a} + \phi(a), \qquad 0 = \lambda - \frac{\mu}{a^2} + \phi'(a). \tag{35}$$

If u is to be harmonic while h_u is expressible in terms of u, u is of the form $\phi(\lambda) + \psi(\mu)$, where $\lambda = x + yi$, $\mu = x - yi$. Since

$$h_u{}^2 = 4 \frac{\partial u}{\partial \lambda} \cdot \frac{\partial u}{\partial \mu},$$

we must have

$$4\phi'(\lambda) \cdot \psi'(\mu) = 4f[\phi(\lambda) + \psi(\mu)], \tag{36}$$

and if we differentiate both sides of this equation with respect to λ and μ we shall get

$$\phi''(\lambda) \cdot \psi'(\mu) = \phi'(\lambda) \cdot f'[\phi(\lambda) + \psi(\mu)],$$
$$\phi'(\lambda) \cdot \psi''(\mu) = \psi'(\mu) \cdot f'[\phi(\lambda) + \psi(\mu)],$$

whence

$$\frac{\phi''(\lambda)}{[\phi'(\lambda)]^2} = \frac{\psi''(\mu)}{[\psi'(\mu)]^2}. \tag{37}$$

Since the first member of (37) involves λ only, and the second member μ only, we may equate each member to a constant, $-k^2$, and consider separately the cases where k is or is not zero.

(1) If $k = 0$, $\phi''(\lambda) = 0$, $\psi''(\mu) = 0$, $\phi(\lambda) = c\lambda + m$,

$$\psi(\mu) = d\mu + n, \qquad \text{and} \qquad u = c\lambda + d\mu + g.$$

If c and $\pm d$ are either real and equal, or conjugate complex quantities, the u curves are a set of real parallel straight lines.

(2) If $k \neq 0$, $\phi'(\lambda) = \dfrac{1}{k^2\lambda + m}$, $\psi'(\mu) = \dfrac{1}{k^2\mu + n}$,

$$\phi(\lambda) = \frac{1}{2^2} \cdot \log(k^2\lambda + m) + a, \qquad \psi(\mu) = \frac{1}{k^2} \cdot \log(k^2\mu + n) + b,$$

[1] Forsyth, *Differential Equations*, p. 307.

and if the constants of integration are so chosen as to make u real or purely imaginary, the u curves are a set of parallel, that is, concentric, circumferences.

Every family of isothermal lines which are the curves of a function u which satisfies (18) is either a set of parallel straight lines or a set of concentric circumferences. No other families of parallel curves are isothermal.

The Equation $h_u \cdot \frac{\partial h_u}{\partial u} = \nabla^2 (u)$.

We have seen that equations (18) and (26) are equivalent; this equation, therefore, defines the families of straight lines which form the orthogonal trajectories of the families of parallel curves defined by the equation $h_v = f(v)$, and we may write

$$q^2 r - 2\,pqs + p^2 t = 0 \tag{38}$$

Monge's method yields the first integral $u = F\left(\frac{p}{q}\right)$, (39)

and of this equation

$$u = \phi\left(\frac{a + y}{c - x}\right) \tag{40}$$

is the complete integral and

$$u = \phi\left(\frac{a + y}{\psi(a) - x}\right), \tag{41}$$

where $\psi(a) - x = (a + y)\,\psi'(a)$, the general integral.

Every family of straight lines in the xy plane, that is every set of lines defined by the equation $ax + py = 1$, where a and p are arbitrary functions of a single parameter, are contained as, of course, they should be, in this general integral.

It is evident that every family of isothermal lines which are the curves of a function u which satisfies (38) is a set of straight lines which pass through a point.

Transformation of the Equation $h_u = f(v)$

Given a function which satisfies (10) or (27), there always exists a function which has the same lines, and a gradient expressible in terms of the orthogonal function alone. The lines of all functions which

satisfy these equations are therefore those of functions which satisfy an equation of the form

$$\frac{\partial h_u}{\partial u} = 0 \tag{42}$$

or

$$p^2r + 2\,pqs + q^2t = 0 \tag{43}$$

If we take advantage of the Principle of Duality and make $p = x'$, $q = y'$, $px + qy - z = z'$, we shall get the transformed equation

$$x'^2 \cdot t' - 2\,x'y' \cdot s' + y'^2 \cdot r' = 0, \tag{44}$$

and if then we put, $m = -2 \log (x^2 + y^2)$, $n = \tan^{-1}\left(\frac{y}{x}\right)$, the result is

$$\frac{\partial u}{\partial m} = \frac{\partial^2 u}{\partial n^2}, \tag{45}$$

which is equivalent to Fourier's familiar equation for the linear flow of heat.

If u is to be harmonic, while $\frac{\partial h_u}{\partial u} = 0$, we may write

$$u = \phi\,(x + yi) + \psi\,(x - yi),$$

and substitute this value in equation (43).

The resulting equation is

$$[\phi'\,(x + yi)]^2 \cdot \psi''\,(x - yi) + [\psi'\,(x - yi)]^2 \cdot \phi''\,(x + yi) = 0, \tag{46}$$

or

$$\frac{\psi''(x - yi)}{[\psi'(x - yi)]^2} = -\frac{\phi''(x + yi)}{[\phi'(x + yi)]^2}. \tag{47}$$

This last equation is possible only if each member is constant, $(-k^2)$, whence $\psi = \frac{1}{k^2} \cdot \log (k^2\mu + m) + a$, $\phi = -\frac{1}{k^2} \cdot \log (k^2\lambda + n) + b$, and the u curves are a family of straight lines meeting in some point. No other families of isothermal lines are possible curves of scalar functions, the gradients of which are expressible in terms of the corresponding orthogonal functions.

III

ON GENERALIZED SPACE DIFFERENTIATION OF THE SECOND ORDER [1]

If one has to investigate the strength of a field of force defined by a given scalar potential function, or to study the flow of electricity in a massive conductor under given conditions, or to apply Green's Theorem to given functions in the space bounded by a given closed surface, or, indeed, to treat any one of a large number of problems in Mathematical Physics or in Analysis, one often needs to find the numerical value at a point, of the derivative of a point function taken in a given direction. This has given rise to the familiar idea of simple space differentiation and of the normal derivative of one scalar function with respect to another; indeed the properties of the first and of the higher space derivatives of a function of n variables taken with respect to any *fixed* direction in n dimensional space, have been treated very clearly and exhaustively by Czuber.[2]

It is sometimes desirable to use also the conception of general space derivatives of the second order. This is the case, for instance, when one is determining the rate of change of the intensity of a conservative field of force at a point which is moving, either along a curved line of force or on a curved surface related to such lines in a prescribed manner. It is easy to define the general space derivative of any order of a given function.

This paper discusses very briefly a few elementary facts with regard to generalized space differentiation of the second order, and treats first, for the sake of simplicity, differentiation of functions of two variables, in the plane of those variables.

Plane Differentiation

Let there be in the xy plane two independent families of curves $(u = c, v = k)$ such that in the domain, R, one and only one curve of

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xxxix, no. 17, February, 1904.

[2] E. Czuber, *Wienerberichte*, p. 1417 (1892).

each family passes through every point, and no curve of either family has anywhere a multiple point. At every point, P, in the domain, the two curves (one of the u family and one of the v family) which pass through the point indicate two directions, s_1, s_2, and if the sense of each of these be determined by any convenient convention, they may be defined by pairs of direction cosines (l_1, m_1), (l_2, m_2), where l_1, m_1, l_2, m_2 are given scalar functions such that at every point

$$l_1^2 + m_1^2 = 1, \quad l_2^2 + m_2^2 = 1. \tag{1}$$

If Ω is any scalar function of the coördinates which within R has finite derivatives of the first and second orders with respect to these coördinates, the derivative of Ω at P in the direction s_1 is the value at the point of the quantity

$$l_1 \cdot \frac{\partial \Omega}{\partial x} + m_1 \cdot \frac{\partial \Omega}{\partial y} \tag{2}$$

and this new scalar function of x and y may be conveniently indicated by the expression $[D_{s_1} \Omega]_P$. If P' is a point on the u curve which passes through P, taken near P and in the sense of the direction s_1, $[D_{s_1} \Omega]_P$ is the limit, as P' approaches P, of $\dfrac{\Omega_{P'} - \Omega_P}{\overline{PP'}}$.

If on the curve of the second (or v) family which passes through P, a point Q be taken near P and in the sense of the direction s_2, the limit, as Q approaches P, of the quantity

$$\frac{[D_{s_1} \Omega]_Q - [D_{s_1} \Omega]_P}{\overline{PQ}} \tag{3}$$

may be indicated by the expression $[D_{s_2} D_{s_1} \Omega]_P$, and this is the second derivative of Ω at P taken with respect to the directions s_1 and s_2 in the order given.

Thus, if

$$\Omega = 2x^2 - y^2, \qquad l_1 = \frac{2x}{\sqrt{4x^2 + 1}}, \qquad m_1 = \frac{1}{\sqrt{4x^2 + 1}},$$

$$l_2 = \frac{y}{\sqrt{x^2 + y^2}}, \quad m_2 = \frac{-x}{\sqrt{x^2 + y^2}}; \qquad D_{s_1} \Omega = \frac{2(4x^2 - y)}{\sqrt{4x^2 + 1}},$$

$$D_{s_2} D_{s_1} \Omega = \frac{x(32x^2y + 8y^2 + 16y + 8x^2 + 2)}{\sqrt{x^2 + y^2} \cdot (4x^2 + 1)^{\frac{3}{2}}}.$$

It is evident from the definition just given that

$$D_{s_2} D_{s_1} \Omega = l_1 \cdot l_2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + (l_1 \cdot m_2 + l_2 \cdot m_1) \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m_1 \cdot m_2 \cdot \frac{\partial^2 \Omega}{\partial y^2} + \left(l_2 \cdot \frac{\partial l_1}{\partial x} + m_2 \cdot \frac{\partial l_1}{\partial y} \right) \cdot \frac{\partial \Omega}{\partial x} + \left(l_2 \cdot \frac{\partial m_1}{\partial x} + m_2 \cdot \frac{\partial m_1}{\partial y} \right) \frac{\partial \Omega}{\partial y}, \qquad (4)$$

and that $D_{s_1} D_{s_2} \Omega$ is quite different in general from $D_{s_2} D_{s_1} \Omega$: the order of the two differentiations is material.

If the u curves happen to be a family of parallel straight lines and the v curves another family of parallel straight lines,

$$D_{s_2} D_{s_1} \Omega = l_1 l_2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + (l_1 m_2 + l_2 m_1) \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m_1 \cdot m_2 \cdot \frac{\partial^2 \Omega}{\partial y^2}, \qquad (5)$$

and the coefficients in this expression are constants.

If the u curves and the v curves are identical and are a family of straight parallel lines, we have

$$D_{s_1}{}^2 \Omega = l_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + 2 l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial y^2}, \qquad (6)$$

the familiar form of the second derivative of Ω along the *fixed direction* s_1, which often appears in work involving the transformation of Cartesian coördinates. Simple special cases of this formula are obtained by putting l equal to 1, 0, and m.

Since $l_1{}^2 + m_1{}^2 = 1$, $\qquad \dfrac{\partial l_1}{\partial x} \cdot \dfrac{\partial m_1}{\partial y} = \dfrac{\partial m_1}{\partial x} \cdot \dfrac{\partial l_1}{\partial y}$,

and if at any point s_1 and s_2 are such as to make the coefficient of $\dfrac{\partial \Omega}{\partial x}$ in (4) vanish, the coefficient of $\dfrac{\partial \Omega}{\partial y}$ will vanish also. Such points as this lie, in general, on a definite curve, the equation of which is to be found by equating one of these coefficients to zero. If s_1 is a fixed direction so that l_1 and m_1 are constants, (4) takes the form (5), but the coefficients are not constants unless s_2 also is fixed.

If the two variable directions s_1, s_2 coincide, (4) becomes the second derivative of the function Ω taken with respect to the direction s_1; that is,

$$D_{s_1}{}^2 \Omega = l_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + 2 l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m^2 \cdot \frac{\partial^2 \Omega}{\partial y^2} + \left(l_1 \cdot \frac{\partial l_1}{\partial x} + m_1 \cdot \frac{\partial l}{\partial y} \right) \frac{\partial \Omega}{\partial x} + \left(l_1 \cdot \frac{\partial m_1}{\partial x} + m_1 \cdot \frac{\partial m_1}{\partial y} \right) \frac{\partial \Omega}{\partial y}. \qquad (7)$$

If the direction cosines of a plane curve at a point on it are l and m, the curvature of the curve at P has the same absolute value as have the expressions

$$\frac{1}{m}\left(l\cdot\frac{\partial l}{\partial x}+m\cdot\frac{\partial l}{\partial y}\right),\quad \frac{1}{l}\left(l\cdot\frac{\partial m}{\partial x}+m\cdot\frac{\partial m}{\partial y}\right). \tag{8}$$

If, therefore, two directions, s_1, s_3, are defined by two curves which, at a point, P, common to both, have a common tangent and equal curvatures, the second derivatives at P of a function Ω taken with respect to the two directions are equal.

If at any point the curvature of the curve of the u family which defines the direction s_1 is zero, the coefficients of $\partial\Omega/\partial x$ and $\partial\Omega/\partial y$ in the expression for $D_{s_1}{}^2\Omega$ at the point vanish. If the u curves are a family of straight lines, the last two terms of (7) disappear, but the coefficients of the other terms are, in general, not constant.

If there is no point in the region R at which both the quantities $\partial\Omega/\partial x$, $\partial\Omega/\partial y$ vanish together, and if the direction s is at every point of R that in which Ω increases most rapidly, $D_s\Omega = h$, where h is the gradient of Ω, that is, the tensor of the gradient vector. Now h is itself, in general, a scalar point function, which, when equated to a parameter, yields a family of curves the directions of which are usually quite different from those of the lines of the gradient-vector. The normal at any point P to the curve of this h family which passes through the point, has the direction cosines

$$\frac{\partial h}{\partial x}\Big/h',\quad \frac{\partial h}{\partial y}\Big/h',$$

where h' is the gradient of h. The angle between the direction, s, of the gradient vector of Ω and the normal to the h curve has at every point the value

$$\cos(\Omega, h) = \left[\frac{\partial h}{\partial x}\cdot\frac{\partial\Omega}{\partial x}+\frac{\partial h}{\partial y}\cdot\frac{\partial\Omega}{\partial y}\right]\Big/hh', \tag{9}$$

and the second derivative of Ω with respect to the direction s is, therefore,

$$D_s{}^2\Omega = \left[\frac{\partial h}{\partial x}\cdot\frac{\partial\Omega}{\partial x}+\frac{\partial h}{\partial y}\cdot\frac{\partial\Omega}{\partial y}\right]\Big/h = h'\cdot\cos(\Omega, h). \tag{10}$$

Let the *normal derivative*,[1] at any point P, of a point function V,

[1] Peirce, *The Newtonian Potential Function*, p. 116; *A Short Table of Integrals*, p. 106.

taken with respect to another point function W, be the limit, as PQ approaches zero, of the ratio of $V_Q - V_P$ to $W_Q - W_P$, where Q is a point so chosen on the normal at P to the surface of constant W which passes through P, that $W_Q - W_P$ is positive: if, then, (V, W) denotes the angle between the directions in which V and W increase most rapidly, the normal derivatives of V with respect to W, and of W with respect to V, may be written

$$[D_W V] = h_V \cdot \cos(V, W)/h_W, \quad [D_V W] = h_W \cdot \cos(V, W)/h_V: \qquad (11)$$

if $h_v = h_w$, these derivatives are equal.

With this notation (10) may be rewritten in the form

$$D_s^2 \Omega = h \cdot [D_\Omega h]. \qquad (12)$$

If at any point $D_s^2\Omega$ vanishes, it is easy to see from (10) that either the gradient (h') of h vanishes at the point, or else the h and Ω surfaces cut each other there orthogonally. This latter case is exemplified in the familiar instance of the electrostatic field due to two long parallel straight wires of the same diameter, charged to equal and opposite potentials: if the wires cut the xy plane normally at P_1, P_2, and if the line joining these intersections be taken for x axis with the point midway between them for origin, the potential function is of the form $V = A \log r_1/r_2$, where $r_1^2 = (x - a)^2 + y^2$, $r_2^2 = (x + a)^2 + y^2$. The intensity of the field, in absolute value, is $h = 2aA/r_1 r_2$, and the second derivative of V taken along the line of force (that is, the rate at which the intensity of the field changes) is numerically equal to

$$\frac{-4aAx}{r_1^2 \cdot r_2^2}.$$

$D_s^2 V$ taken along a line of force vanishes, therefore, at all points on the y axis, and at all such points the curve of constant $V(r_1/r_2 = b)$ cuts the curves of constant $h(r_1 r_2 = k)$ orthogonally. At points on the y axis the direction of the lines of force is parallel to the x axis, and the second derivative of V with respect to the fixed direction x happens to vanish here also where $l = 1$, $\frac{\partial l}{\partial x} = 0$, $m = 0$, $\frac{\partial V}{\partial y} = 0$. The quantity h' does not vanish at any finite point.

The example just discussed is in contrast with the case where the Ω family are a set of parallel curves of any kind, and h in consequence

(if not constant) is a function of Ω alone, so that the h curves and the Ω curves coincide, and if $D_s{}^2\Omega$ vanishes anywhere, it must be where h' vanishes. A simple example of this is furnished by the field of attraction within a very long cylinder of revolution, the density of which is a function of the distance from the axis alone.

If the directions s_1 and s_2 are everywhere perpendicular to each other, we may without loss of generality write $l_2 = -m_1$, $m_2 = l_1$; in which case the coefficients of $\partial\Omega/\partial x$, $\partial\Omega/\partial y$ in (4) become

$$\left(l_2 \cdot \frac{\partial m_2}{\partial x} + m_2 \cdot \frac{\partial m_2}{\partial y}\right) \text{ and } -\left(l_2 \cdot \frac{\partial l_2}{\partial x} + m_2 \cdot \frac{\partial l_2}{\partial y}\right): \tag{13}$$

these vanish if the v curves form a family of straight lines, or the u curves a family of straight or curved parallels. The order of differentiation with respect to the orthogonal directions s_1, s_2 is immaterial if both the u and the v curves are straight lines, that is, if the directions are fixed.

If s_1 is the direction in which Ω increases most rapidly, and s_2 the direction of constant Ω,

$$D_{s_2}D_{s_1}\Omega = D_{s_2}h = \left[\frac{\partial\Omega}{\partial x}\cdot\frac{\partial h}{\partial y} - \frac{\partial\Omega}{\partial y}\cdot\frac{\partial h}{\partial x}\right]\Big/ h$$

$$= \left\{\frac{\partial^2\Omega}{\partial x\cdot\partial y}\left[\left(\frac{\partial\Omega}{\partial x}\right)^2 - \left(\frac{\partial\Omega}{\partial y}\right)^2\right] + \frac{\partial\Omega}{\partial x}\cdot\frac{\partial\Omega}{\partial y}\left[\frac{\partial^2\Omega}{\partial y^2} - \frac{\partial^2\Omega}{\partial x^2}\right]\right\}\Big/ h. \tag{14}$$

Now the direction cosines and the slope of the line of the gradient vector at any point are

$$\frac{1}{h}\cdot\frac{\partial\Omega}{\partial x}, \quad \frac{1}{h}\cdot\frac{\partial\Omega}{\partial y}, \text{ and } \frac{\partial\Omega}{\partial y}\Big/\frac{\partial\Omega}{\partial x}.$$

So that the curvature of the line is

$$\frac{1}{\rho} = \frac{\dfrac{d}{dx}\left[\dfrac{\partial\Omega}{\partial y}\Big/\dfrac{\partial\Omega}{\partial x}\right]}{\left[1 + \left(\dfrac{\partial\Omega}{\partial y}\Big/\dfrac{\partial\Omega}{\partial x}\right)^2\right]^{\frac{3}{2}}}$$

$$= \left\{\frac{\partial^2\Omega}{\partial x\cdot\partial y}\left[\left(\frac{\partial\Omega}{\partial x}\right)^2 - \left(\frac{\partial\Omega}{\partial y}\right)^2\right] + \frac{\partial\Omega}{\partial x}\cdot\frac{\partial\Omega}{\partial y}\left[\frac{\partial^2\Omega}{\partial y^2} - \frac{\partial^2\Omega}{\partial x^2}\right]\right\}\Big/ h^3 \tag{15}$$

and we may write in this case

$$D_{s_2}D_{s_1}\Omega = h/\rho.$$

This expression gives the rate at which the maximum slope of the surface the coördinates of which are (x, y, Ω), changes as one goes along a line of level.[1]

When s_1 and s_2 are perpendicular to each other, we have in general

$$D_{s_2}{}^2\Omega = m_1{}^2 \cdot \frac{\partial^2\Omega}{\partial x^2} - 2l_1m_1 \cdot \frac{\partial^2\Omega}{\partial x \cdot \partial y} + l_1{}^2 \cdot \frac{\partial^2\Omega}{\partial y^2} + \left(m_1 \cdot \frac{\partial m_1}{\partial x} - l_1 \cdot \frac{\partial m_1}{\partial y}\right)\frac{\partial\Omega}{\partial x} + \left(l_1 \cdot \frac{\partial l_1}{\partial y} - m_1 \cdot \frac{\partial l_1}{\partial x}\right)\frac{\partial\Omega}{\partial y}, \quad (16)$$

and since $l_1{}^2 + m_1{}^2 = 1$,

$$l_1 \cdot \frac{\partial l_1}{\partial x} + m_1 \cdot \frac{\partial m_1}{\partial x} = 0, \qquad l_1 \cdot \frac{\partial l_1}{\partial y} + m_1 \cdot \frac{\partial m_1}{\partial y} = 0.$$

So that if we add together (7) and (16) we shall get

$$D_{s_1}{}^2\Omega + D_{s_2}{}^2\Omega = \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{1}{m_1} \cdot \frac{\partial l_1}{\partial y} \cdot \frac{\partial\Omega}{\partial x} + \frac{1}{l_1} \cdot \frac{\partial m_1}{\partial x} \cdot \frac{\partial\Omega}{\partial y}$$
$$= \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} - \frac{1}{l_1} \cdot \frac{\partial m_1}{\partial y} \cdot \frac{\partial\Omega}{\partial x} - \frac{1}{m_1} \cdot \frac{\partial l_1}{\partial x} \cdot \frac{\partial\Omega}{\partial y}. \quad (16')$$

It is evident that the values of the space derivatives defined above are wholly independent of the particular system of rectangular coordinates which may be used.

Space Differentiation

At every point of the space domain, R, let two independent directions (s_1, s_2) be defined by the direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), where $l_1, m_1, n_1, l_2, m_2, n_2$ are any six single-valued point functions which satisfy the identities

$$l_1{}^2 + m_1{}^2 + n_1{}^2 = 1, \qquad l_2{}^2 + m_2{}^2 + n_2{}^2 = 1, \quad (17)$$

and have finite derivatives of the first order with respect to the coordinates x, y, z. If, then, Ω is any single-valued function of the coordinates which within R has finite derivatives of the first and second orders with respect to these coördinates, the derivative of Ω at the point P, in the direction s_1, is the value at P of the quantity

[1] Boussinesq, *Cours d'Analyse Infinitésimale*, t. i, f. 2, p. 236.

$$D_{s_1}\Omega \equiv l_1 \cdot \frac{\partial \Omega}{\partial x} + m_1 \cdot \frac{\partial \Omega}{\partial y} + n_1 \cdot \frac{\partial \Omega}{\partial z} \cdot \tag{18}$$

Through the point P passes a curve of the family defined by the equations

$$\frac{dx}{l_2} = \frac{dy}{m_2} = \frac{dz}{n_2}, \tag{19}$$

and this curve indicates the direction s_2. If on this curve a point Q be taken near P and in the sense of the direction s_2, the limit, as Q approaches P, of the quantity

$$\frac{[D_{s_1}\Omega]_Q - [D_{s_1}\Omega]_P}{\overline{PQ}} \tag{20}$$

may be represented by $[D_{s_2}D_{s_1}\Omega]_P$ and this is the second directional derivative at P of Ω taken with respect to the directions s_1 and s_2 in the order given. It is evident that

$$\begin{aligned} D_{s_2}D_{s_1}\Omega = {} & l_1 l_2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + m_1 m_2 \cdot \frac{\partial^2 \Omega}{\partial y^2} + n_1 n_2 \cdot \frac{\partial^2 \Omega}{\partial z^2} \\ & + (l_1 m_2 + l_2 m_1) \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + (m_1 n_2 + m_2 n_1) \frac{\partial^2 \Omega}{\partial y \cdot \partial z} + (n_1 l_2 + n_2 l_1) \frac{\partial^2 \Omega}{\partial z \cdot \partial x} \\ & + \left(l_2 \cdot \frac{\partial l_1}{\partial x} + m_2 \cdot \frac{\partial l_1}{\partial y} + n_2 \cdot \frac{\partial l_1}{\partial z} \right) \frac{\partial \Omega}{\partial x} \\ & + \left(l_2 \cdot \frac{\partial m_1}{\partial x} + m_2 \cdot \frac{\partial m_1}{\partial y} + n_2 \cdot \frac{\partial m_1}{\partial z} \right) \frac{\partial \Omega}{\partial y} \\ & + \left(l_2 \cdot \frac{\partial n_1}{\partial x} + m_2 \cdot \frac{\partial n_1}{\partial y} + n_2 \cdot \frac{\partial n_1}{\partial z} \right) \frac{\partial \Omega}{\partial z}, \end{aligned} \tag{21}$$

and that this is not equal to $D_{s_1}D_{s_2}\Omega$.

If the directions s_1, s_2 are fixed, the six direction cosines are constants, the last three terms of (21) disappear, and the coefficients of the other six terms are constant. If the fixed directions s_1, s_2 coincide, (21) reduces to the familiar form

$$\begin{aligned} D_{s_1}^2\Omega = {} & l_1^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + m_1^2 \cdot \frac{\partial^2 \Omega}{\partial y^2} + n_1^2 \cdot \frac{\partial^2 \Omega}{\partial z^2} \\ & + 2\, l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + 2\, m_1 n_1 \cdot \frac{\partial^2 \Omega}{\partial y \cdot \partial z} + 2\, l_1 n_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial z}, \end{aligned} \tag{22}$$

whereas, if s_1 is not fixed,

$$D_{s_1}{}^2\Omega = l_1{}^2 \cdot \frac{\partial^2\Omega}{\partial x^2} + m_1{}^2 \cdot \frac{\partial^2\Omega}{\partial y^2} + n_1{}^2 \cdot \frac{\partial^2\Omega}{\partial z^2} + 2l_1m_1 \cdot \frac{\partial^2\Omega}{\partial x \cdot \partial y} + 2m_1n_1 \cdot \frac{\partial^2\Omega}{\partial y \cdot \partial z}$$
$$+ 2l_1n_1 \cdot \frac{\partial^2\Omega}{\partial x \cdot \partial z} + \left(l_1 \cdot \frac{\partial l_1}{\partial x} + m_1 \cdot \frac{\partial l_1}{\partial y} + n_1 \cdot \frac{\partial l_1}{\partial z}\right)\frac{\partial\Omega}{\partial x}$$
$$+ \left(l_1 \cdot \frac{\partial m_1}{\partial x} + m_1 \cdot \frac{\partial m_1}{\partial y} + n_1 \cdot \frac{\partial m_1}{\partial z}\right)\frac{\partial\Omega}{\partial y}$$
$$+ \left(l_1 \cdot \frac{\partial n_1}{\partial x} + m_1 \cdot \frac{\partial n_1}{\partial y} + n_1 \cdot \frac{\partial n_1}{\partial z}\right)\frac{\partial\Omega}{\partial z}. \qquad (23)$$

All the coefficients in (22) are constants; all those of (23) are in general variable. If s_1 is defined by any infinite system of straight lines of which just one passes through every point of space, and if the direction s_1 at all points of any one of the lines is that of the line itself, the coefficients of $\partial\Omega/\partial x$, $\partial\Omega/\partial y$, $\partial\Omega/\partial z$ in (23) vanish. In particular, if the direction s_1 is that of the radius vector from a fixed point (a, b, c), (23) takes the form of (22) though the remaining coefficients are not constants. In any case if the coefficients of two of the three quantities $\partial\Omega/\partial x$, $\partial\Omega/\partial y$, $\partial\Omega/\partial z$ vanish, the third must vanish also.

If the gradient, h, of Ω does not vanish at any point of R and if s is the direction in which Ω increases most rapidly,

$$D_s\,\Omega = h,$$
$$D_s{}^2\Omega = \left[\frac{\partial h}{\partial x} \cdot \frac{\partial\Omega}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial\Omega}{\partial y} + \frac{\partial h}{\partial z} \cdot \frac{\partial\Omega}{\partial z}\right] \Big/ h. \qquad (24)$$

If h' is the gradient of the scalar point function which gives the value of h, and if (Ω, h) represents the angle between the directions in which the point functions Ω and h increase most rapidly,

$$\cos(\Omega, h) = \left[\frac{\partial h}{\partial x} \cdot \frac{\partial\Omega}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial\Omega}{\partial y} + \frac{\partial h}{\partial z} \cdot \frac{\partial\Omega}{\partial z}\right] \Big/ h \cdot h' \qquad (25)$$

and
$$D_s{}^2\Omega = h' \cdot \cos(\Omega, h), \text{ or } h\,[D_\Omega h] \qquad (26)$$

where $[D_\Omega h]$ represents the normal derivative of h with respect to Ω.

If the equation $\Omega = k$ happens to represent a set of parallel surfaces, h, if not constant, is a function of Ω alone, so that the h and Ω surfaces are coincident: in this case $\cos(\Omega, h) = 1$ and $D_s{}^2\Omega$ can van-

ish only where h' vanishes. In general, $D_s{}^2\Omega$ vanishes when the h and Ω surfaces cut each other at right angles.

If s_1, s_2, s_3 are any three mutually perpendicular directions,

$$l_1{}^2 + l_2{}^2 + l_3{}^2 = m_1{}^2 + m_2{}^2 + m_3{}^2 = n_1{}^2 + n_2{}^2 + n_3{}^2 = 1,$$

$$l_1m_1 + l_2m_2 + l_3m_3 = m_1n_1 + m_2n_2 + m_3n_3 = l_1n_1 + l_2n_2 + l_3n_3 = 0,$$

and
$$D_{s_1}{}^2\Omega + D_{s_2}{}^2\Omega + D_{s_3}{}^2\Omega = \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2}$$

$$+ \frac{\partial\Omega}{\partial x}\left[m_1 \cdot \frac{\partial l_1}{\partial y} + m_2 \cdot \frac{\partial l_2}{\partial y} + m_3 \cdot \frac{\partial l_3}{\partial y} + n_1 \cdot \frac{\partial l_1}{\partial z} + n_2 \cdot \frac{\partial l_2}{\partial z} + n_3 \cdot \frac{\partial l_3}{\partial z}\right]$$

$$+ \frac{\partial\Omega}{\partial y}\left[n_1 \cdot \frac{\partial m_1}{\partial z} + n_2 \cdot \frac{\partial m_2}{\partial z} + n_3 \cdot \frac{\partial m_3}{\partial z} + l_1 \cdot \frac{\partial m_1}{\partial x} + l_2 \cdot \frac{\partial m_2}{\partial x} + l_3 \cdot \frac{\partial m_3}{\partial x}\right]$$

$$+ \frac{\partial\Omega}{\partial z}\left[l_1 \cdot \frac{\partial n_1}{\partial x} + l_2 \cdot \frac{\partial n_2}{\partial x} + l_3 \cdot \frac{\partial n_3}{\partial x} + m_1 \cdot \frac{\partial n_1}{\partial y} + m_2 \cdot \frac{\partial n_2}{\partial y} + m_3 \cdot \frac{\partial n_3}{\partial y}\right]. \quad (27)$$

IV

ON THE PROPERTIES OF MAGNETS MADE OF HARDENED CAST IRON [1]

During the last six or seven years a large number of d'Arsonval galvanometers, in which the permanent fields are due to hardened and artificially seasoned cast-iron magnets, have been used in the Physical Laboratory of Harvard University, in competition with similar instruments furnished with hardened forged-steel magnets from the shops of well-known makers. For nearly five years also magnets of the same kind have been employed in standard mirror amperemeters and voltmeters fixed in the laboratory, in cases where it was desirable that the indications of the instruments should be trustworthy within one part in a thousand of their larger deflections, over a considerable range of room temperatures. Besides the cast-iron magnets which we have made ourselves, we have of late used a number of others in moving-coil galvanometers purchased in the market — some of the best of them from the Leeds and Northrup Company.

It early appeared from tests made on these instruments, that whereas good iron castings as they come from the foundry make most unsatisfactory magnets, so far as permanence is concerned, magnets made of castings properly hardened and aged after being machined — if machining is necessary for the purpose to which the magnets are to be put — compare favorably in strength, in permanence, and in the relatively small changes of their moments with room temperature, with the best of tool steel magnets, even if in strength, though not in their other qualities, they fall a little behind magnets made, in a forming press, of steel specially prepared for the purpose.

Although chilled cast-iron bar magnets have been used for a long time in a few forms of telephones, it is usually best to make straight magnets (which do not need to be hammered), of steel; but the forging of steel for permanent magnets of complex forms, without spoiling

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xl, no. 22, April, 1905.

it, demands a kind of skill which most toolmakers, even in the largest establishments, have not acquired, and it is generally difficult to get satisfactory specimens of any very special shape of curved-steel magnets unless one has access to such facilities as a few of the manufacturers of electrical measuring instruments have provided for themselves. It is true that the hardening of iron castings for magnetic purposes also requires such skill as few persons possess, if the very best results are to be obtained, especially when the pieces to be treated weigh more than a pound or two; but a little practice will enable any good workman, who has a gas forge with blast powerful enough to raise the temperature of the iron uniformly nearly to the melting-point, to make good gray-iron castings of moderate size, hard enough for strong magnets, which will leave little to be desired so far as permanence is concerned. It has been my good fortune to have the help of Mr. G. W. Thompson, the mechanician of the Jefferson Laboratory, who has had long experience in treating cast iron, and who has made for me, by a process of his own, massive magnets with extremely low temperature coefficients. It is to be noticed that some of the secret methods of hardening cast iron, used by makers of small parts of machinery, do not fit the castings for making good magnets, and that case hardening, which affects the surface only, is useless. The character of the cold bath into which, while it is kept in violent agitation, the strongly heated castings to be hardened are plunged, seems to have considerable influence upon the result.

At the very high temperature, just under the melting-point, to which the cast iron must be raised before it is suddenly chilled, the metal loses much of its tenacity, and slender pieces must be handled carefully lest they break like chalk. The chilled casting should be hard enough to scratch window glass, if not so readily as hardened tool steel will do it. It is vain to attempt to make any such gray-iron castings as I have used, magnetically hard by chilling them after they have been heated to the comparatively low temperatures that one would use in making steel glass-hard. Everyone who has attempted to harden a thick mass of tool steel uniformly, knows how difficult the task is: it is easy enough to get the outer layers glass-hard, while the interior is much softer; or, sometimes (by overheating the steel), to get the inside hard while the outside is blistered and cracked. If

a casting is heated to a very bright red and then plunged into the bath, the outside may become hard to the file, while the interior, as magnetic tests clearly show, remains soft; in this case, however, the material will stand a higher temperature without injury, and if the mass be reheated and when it is just below the melting-point be suddenly chilled, the whole interior becomes hard.

It is a good deal easier to harden a lot of straight, round pieces of good gray cast iron, say 20 centimeters long and 1 centimeter in diameter, so that they shall all be nearly alike magnetically, than it is to do the same with an equal number of pieces of drill rod. Six pieces of Crescent Drill Rod each 16 centimeters long and 8 millimeters in diameter, cut from the same specimen, were made glass-hard for me by a skilled worker in steel; these were placed successively in a properly oriented solenoid and exposed, first to the action of an alternating current of intensity gradually decreasing from an initially high value to a very low one, then to a steady field of 147 gausses applied first in one direction and afterwards in the other. As a consequence of the preliminary treatment [1] with alternating currents, the magnitudes of the moments acquired by the pieces under the action of the steady field were quite independent of the direction of the latter. These moments were approximately 2280, 2395, 2495, 2326, 2325, and 2360, but when the field was removed the residual moments were 1058, 1074, 1136, 1066, 1050, and 1097 respectively. The same pieces were then placed together in a solenoid made of many turns of large wire and the ends of the whole bundle were connected by a massive yoke; when a current of about 45 amperes was sent through the wire the pieces became charged practically to saturation. When they were removed from the solenoid the average moment of the six was about 1240, the highest 1290, and the lowest 1170. Such uniformity as is indicated by these numbers is, I believe, as great as one can expect to get unless one has an elaborate plant; no such agreement can be hoped for from pieces of different rods of the same brand. Some kinds of special magnet steel give rather better results.

Although it is obvious that there is no advantage in using cast-iron for straight magnets, I have had a number of such magnets made, of each of three shapes, for purposes of comparison with steel bar mag-

[1] J. A. Ewing, *Magnetic Induction in Iron and other Metals*, 1892, p. 328.

nets of the same dimensions. These were all rather short, because we had no means of treating satisfactorily very long, slender pieces, which are apt to warp if not properly supported. It is, of course, impossible to calculate the demagnetizing effects of the free ends of such pieces as I have used, but it has seemed to me legitimate to draw some inferences from the hysteresis curves and from the temperature coefficients of rods of different materials if they are geometrically alike.

Several years ago, when I had to have a set of the best magnets I could get for measuring purposes, carefully ground into shape after the steel had been hardened, I experimented [1] upon hundreds of seasoned magnets made of many kinds of steel; ordinary tool steels, self-hardening tool steels, and special magnet steels. In comparing round steels, it appeared that such "Stub's Polished Drill Rod" as I could get made slightly less desirable magnets than did "Crescent Drill Rod," or the common brand of "Jessop's Black Rod," which last two were magnetically indistinguishable. I found nothing better in tool steels than the Crescent Drill Rod, or the Jessop's Black Round Tool Steel, and I have used these as standards in testing my round cast-iron magnets. The subjoined table shows what may be expected of seasoned magnets made of these steels.

Most of my experiments on the strengths of round cast-iron bar magnets have been made with pieces 18 centimeters long and either 0.95 centimeters or 1.25 centimeters in diameter, of which I have a good many, some new and some cast two years ago. These were usually relaxed [2] after their hardening by being boiled in water for some time; next they were magnetized to saturation in a solenoid, and then they were again boiled and "aged" in the usual manner. The resulting magnets were finally tested in competition with a large number of seasoned tool steel magnets, of different brands but all of the same dimensions as the castings, with the help of a mirror magnetometer. The cast-iron magnets looked, of course, rather rough in comparison with others made of polished rod, but their moments differed among themselves less than those of an equal number of the steel magnets made of any one brand. Just one of the tool steel

[1] Peirce, *American Journal of Science*, 1898, p. 334.
[2] Barus and Strouhal, *Bulletin of the U. S. Geological Survey*, no. 14, 1885.

RESULTS OF EXPERIMENTS MADE UPON PIECES OF CRESCENT DRILL ROD AND JESSOP'S BLACK ROUND TOOL STEEL

Length in centimeters	Diameter in centimeters	Weight in grams per centimeter of length	Moment per gram of magnetism induced in the specimen when placed lengthwise in a unit field				Permanent magnetic moment per gram of the seasoned magnet	Approximate ratio of the induced moment caused by unit field, to the permanent moment in the seasoned magnet	Mean loss per degree of the magnetic moment when the magnet is heated from about 15° C. to 100° C., expressed in terms of the moment at the lower temperature	Temperature coefficient of the seasoned magnet at 15° C.
			In the condition in which it was purchased	When "glass-hard" but still unmagnetized	When magnetized to saturation while "glass-hard"	In the condition of a seasoned magnet				
20	1.11	7.6	0.44	0.23	0.28	0.30	30.5	0.010	0.00048	0.00047
18			0.41	0.22	0.26	0.28	25.5	0.011	0.00049	0.00048
16			0.38	0.21	0.24	0.25	21.5	0.012	0.00051	0.00050
14			0.34	0.20	0.22	0.23	17.5	0.013	0.00053	0.00052
12			0.29	0.19	0.19	0.20	15.0	0.014	0.00057	0.00057
10			0.25	0.17	0.17	0.18	12.0	0.015	0.00062	0.00062
8			0.20	0.15	0.14	0.15	10.0	0.015	0.00070	0.00070
6			0.15	0.13	0.11	0.11	8.5	0.015	0.00088	0.00087
4			0.11	0.08	0.08	0.08	5.5	0.017	0.00110	0.00109
20	0.95	5.5	0.50	0.23	0.25	0.26	36.5	0.007	0.00040	0.00036
18			0.48	0.23	0.24	0.25	32.5	0.008	0.00046	0.00041
16			0.45	0.22	0.23	0.24	28.0	0.009	0.00051	0.00046
14			0.42	0.21	0.21	0.22	24.0	0.009	0.00056	0.00051
12			0.37	0.20	0.19	0.20	20.5	0.010	0.00063	0.00056
10			0.31	0.18	0.17	0.18	17.0	0.011	0.00071	0.00063
8			0.25	0.16	0.15	0.16	14.0	0.012	0.00078	0.00070
6			0.19	0.13	0.13	0.13	10.0	0.013	0.00086	0.00078
4			0.12	0.09	0.10	0.10	7.0	0.014	0.00095	0.00087
20	0.80	3.9	0.61	0.24	0.24	0.30	41.0	0.007	0.00040	0.00037
18			0.58	0.23	0.23	0.29	36.5	0.008	0.00047	0.00042
16			0.54	0.22	0.22	0.28	32.5	0.009	0.00053	0.00047
14			0.49	0.20	0.20	0.27	26.5	0.010	0.00061	0.00052
12			0.44	0.18	0.18	0.26	24.0	0.011	0.00066	0.00058
10			0.39	0.16	0.16	0.24	20.0	0.012	0.00073	0.00064
8			0.31	0.14	0.14	0.21	17.0	0.012	0.00080	0.00070
6			0.23	0.12	0.12	0.16	13.0	0.014	0.00087	0.00077
4			0.15	0.09	0.09	0.12	8.5	0.014	0.00094	0.00083
20	0.64	2.4	0.69	0.26	0.23	0.25	43.0	0.006	0.00024	0.00020
18			0.65	0.25	0.23	0.24	38.0	0.006	0.00025	0.00020
16			0.61	0.24	0.22	0.24	33.5	0.007	0.00026	0.00021
14			0.56	0.23	0.21	0.23	28.0	0.008	0.00027	0.00021
12			0.49	0.22	0.20	0.22	23.0	0.009	0.00030	0.00023
10			0.42	0.21	0.19	0.21	19.0	0.010	0.00035	0.00026
8			0.33	0.19	0.17	0.18	15.0	0.012	0.00043	0.00030
6			0.25	0.16	0.13	0.15	11.0	0.014	0.00060	0.00039
4			0.16	0.11	0.10	0.10	7.0	0.014	0.00091	0.00049

magnets had a moment sensibly greater (about 4 per cent) than any of the cast-iron magnets, but the average of the moments of the cast-iron magnets was rather greater than those of the steel, even after the records of the two or three steel magnets with abnormally low moments had been rejected.

Two years ago[1] I measured the temperature coefficients of a good many bar magnets of cast iron and steel. In every instance, as was to be expected, the rate of loss of moment per degree of rise of temperature was much greater at temperatures near the boiling-point of water than at room temperatures, but if for purposes of comparison we used the mean loss, per degree, of the magnetic moment, when the magnet was heated from about 10° C. to 100° C., expressed in terms of the moment at the lower temperature, it appeared that in the case of rods 16 cm. long and 0.95 cm. in diameter these mean losses were

0.00042 for seasoned, chilled, cast-iron magnets,
0.00046 for seasoned magnets made of Crescent Steel Drill Rod,
0.00046 for seasoned magnets made of Jessop's Round Black Tool Steel,

while they were

0.00070 for seasoned magnets made of Jessop's Square Tool Steel 16 cm. long and of cross-section nearly that of the round rods.

In the case of shorter rods the difference was still more in favor of the castings because, I suppose, they were more uniformly hardened in the interior than the steel could be.

If an iron casting which has been hardened and boiled is magnetized in a solenoid either to saturation or to a degree which falls much short of this, it is practically impossible to decrease the moment by even so little as a tenth of one per cent, by striking the magnet on end with a wooden mallet or with a stone. I have tested many such magnets by dropping each two or three hundred times upon a stone slab, or by giving it hundreds of sharp blows with wooden clubs: the magnets get a little warm during this harsh treatment, but when their temperatures again fall to the original point the moments, which may have fallen a small fraction of one per cent, regain wholly, so far as my observations go, their old values. Some few of the specimens of special magnet steel that I have examined are nearly equal to the castings in this respect.

[1] Peirce, *Proceedings of American Academy*, February, 1903.

Prolonged boiling has, however, always reduced the moment of the cast-iron magnets very sensibly, and this loss may be as much as 20 per cent when the magnetizing field has been an extremely strong one and the residual moment is very high; if the casting has not been magnetized to saturation, the loss of moment by boiling is much less. If, after a cast-iron magnet has been seasoned, its temperature be suddenly raised from room temperature to 100° C., and then as suddenly lowered, the magnet may not wholly recover its original strength until after the lapse of several hours; if, however, the upper limit be only 50° C., there seems to be no sensible lag in the attainment of the whole of the original moment after the testing.

Most of the steel made specially for magnets, which I have been able to get, has come in the form of long bars of rectangular cross-section, about 2 cm. wide and 1 cm. thick; of such steel I have had pieces, 18 cm. long, hardened in some of the various ways used by professional magnet-makers, to compare with hardened castings of the same dimensions. One of the hardened but unmagnetized castings was placed in a solenoid, exposed to the action of a demagnetizing alternating current, and then put rapidly two or three times through a hysteresis cycle, using a maximum field of 145.2 gausses. After this preliminary treatment — which I used in the case of every piece that I examined — the hysteresis diagram remained unchanged however many times the iron went through the cycle. I first obtained twenty-two points on half the diagram, laid those down carefully upon a piece of coördinate paper on such a scale that the diagram was 41 cm. long, and found that it was possible to draw a smooth curve which would not lie away from any one of the points by so much as a quarter of a millimeter and would apparently pass through almost every point; then I completed the cycle and found that the final reading of the mirror magnetometer did not differ by more than a sixth of one per cent, if by so much, from the initial reading. In the course of its second journey around the cycle, the iron made subsidiary loops on opposite sides of the origin and then returned again sensibly to its original condition. No diagram obtained from a long, slender wire could have been smoother than this one which belonged to this short bar. All the hysteresis curves in this paper are reduced from large drawings; generally about twenty-four points (though sometimes

more) were found for each half diagram, and the curve was drawn through practically all of these.

Figure 1 shows an instance where the points were apparently not so well determined as in other cases. This figure represents hysteresis curves of (A) a piece of hardened cast iron (18 cm. × 2 cm. × 1 cm.), and (B) of two pieces of Seebohm and Dickstahl magnet steel (of the

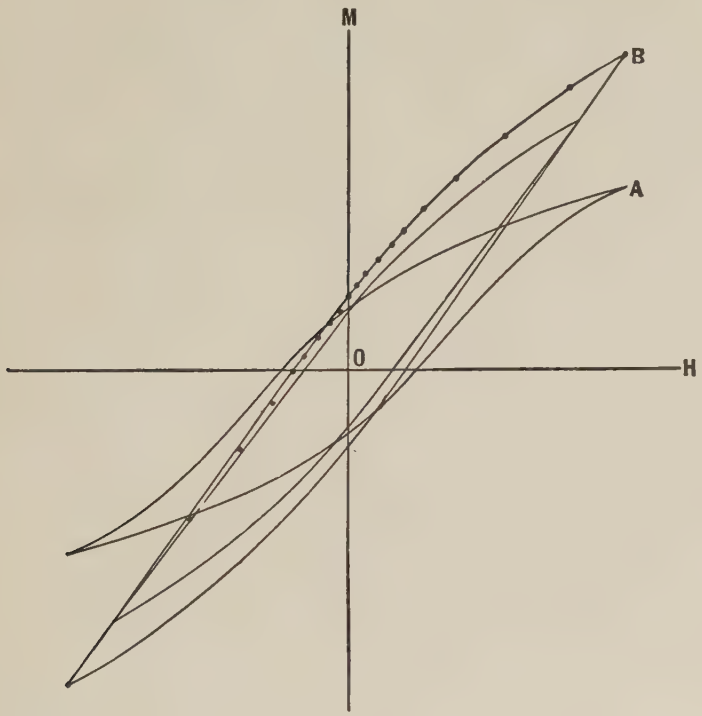


FIGURE 1

same dimensions), which makes the strongest saturated magnets of any of the special steels which I have used. The maximum field in the case of the A curve and the larger B curve was nearly 166 gausses, and under this field the cast iron and the special steel acquired moments of about 8900 and 15400 units respectively: when the field was removed the residual moments were about 3120 and 3550. The piece of steel just mentioned was hardened in a water bath; the smaller B diagram was obtained at another time with a similar piece

of the same steel chilled in one of the baths used by Mr. Thompson. Although the maximum fields were unfortunately not the same in the two B diagrams, the general shapes of the curves are very similar.

These two pieces of steel and the cast iron were then placed in the solenoid mentioned above, the ends of each piece were connected together outside the coil by a massive iron yoke, and a current of about

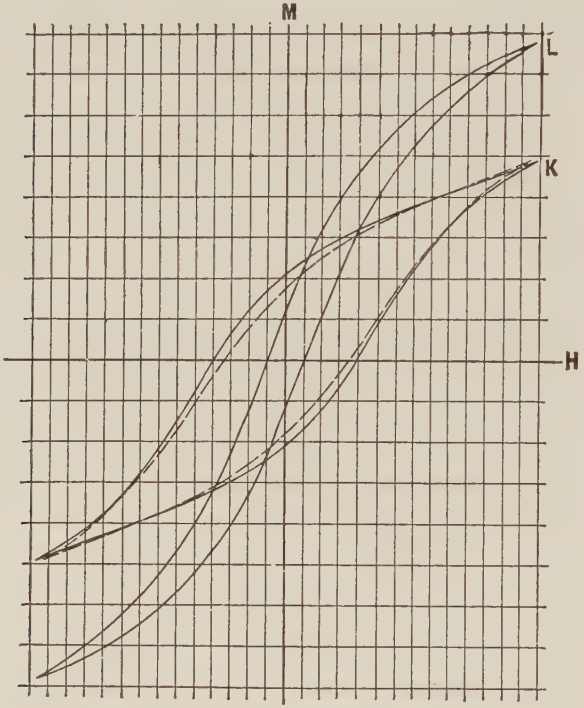


FIGURE 2

45 amperes was sent through the coil; the residual moments of the cast iron, the steel chilled in water, and the steel hardened in the special bath were then about 4390, 5500, and 6220. The weight of each piece of steel was approximately 274 grams. Of three other pieces of steel of a brand used exclusively by well-known makers of magnets for "magnetos," the first was hardened in plain water, the second treated with potassium ferrocyanide, the third chilled in Mr. Thompson's bath. These pieces, like those just described, were 18

cm. long, 2 cm. wide, and 1 cm. thick, and they retained, after being magnetized to saturation, the moments 3470, 2500, and 4190 approximately: this steel was in our hands, therefore, not so good as the Seebohm and Dickstahl Special. The special magnet steels can be bent or formed, but cannot be heated hot enough for welding without spoiling the material for making magnets.

In Figure 2 the results of observations made by Mr. John Coulson and myself upon round cast-iron rods 18 cm. long and 0.95 cm. in diameter are shown. The rods were magnetized in a solenoid, of 4927 turns in a length of 980 millimetres, placed with its axis horizontal and perpendicular to the meridian, and the relative moments, at different times, of the rod under investigation were determined from the deflection of a mirror magnetometer placed outside the solenoid in the Second Position of Gauss with respect to the magnet. The abscissas represent the field (in tens of gausses) to which the rod is subjected by the current in the solenoid; the ordinates represent the corresponding magnetic moment of the rod in thousands of units. The L-curve was obtained with a soft casting, the heavy K-curve with a casting of the same lot which had been chilled from a high temperature in cold water. At the same time, with this casting, were hardened two or three pieces of Crescent Drill Rod of the same dimensions as it. These were naturally not exactly alike magnetically, but the best of them furnished the dotted K-curve of Figure 2; the others had a very little less retentiveness. The magnetic likeness of the cast iron and the steel is worthy of notice.

In Figure 3 the horizontal divisions represent, as before, the field in tens of gausses, the vertical divisions the corresponding moments of the specimens in thousands of units.

The C-diagram was obtained with a soft casting 18 cm. long and 0.95 cm. in diameter, the D-curve with a casting of the same lot hardened by Mr. Thompson's methods. Another casting of the same set, hardened at the same time, gave the continuous curve in Figure 4, while a piece of Crescent Drill Rod, treated with the castings, furnished the dotted curve in the same figure. A careful comparison of the curves of Figures 3 and 4, obtained with the chilled castings — which were chosen from the lot wholly at random — will show that they are magnetically almost indistinguishable; the likeness of the Cres-

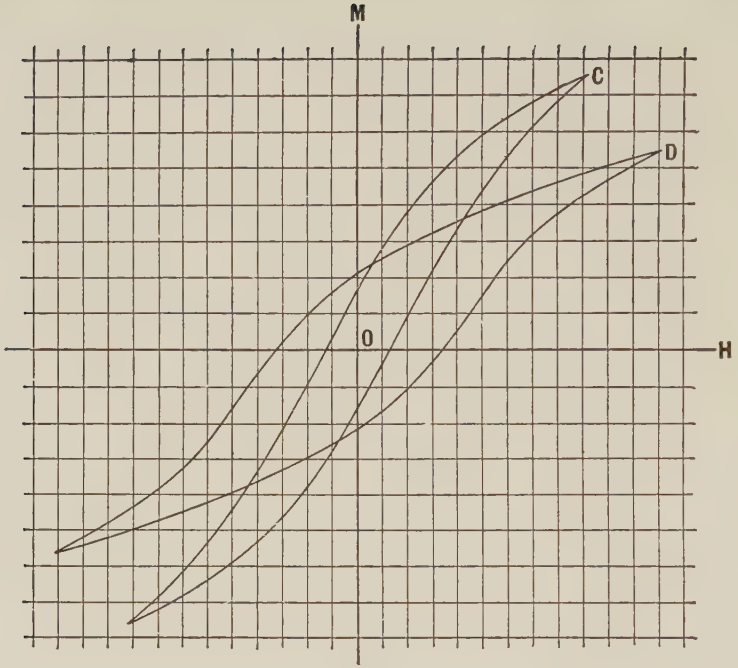


FIGURE 3

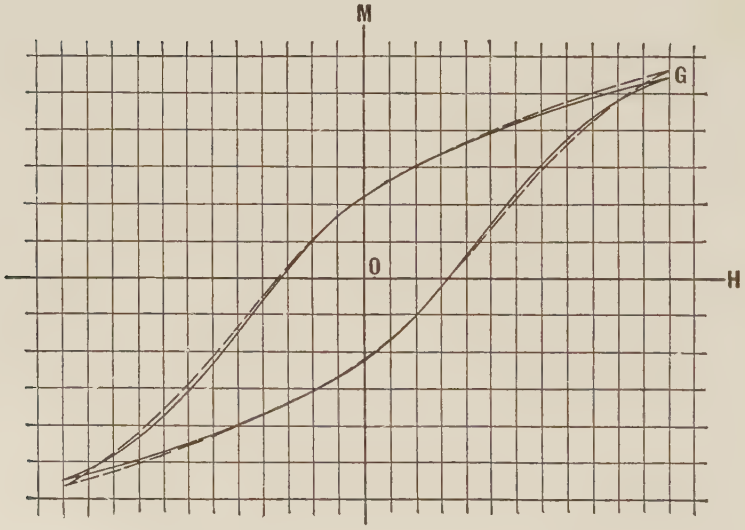


FIGURE 4

cent Drill Rod, chosen also at random, to the castings is again shown by the curves of Figure 4.

We have used cast-iron magnets of many different forms, a few of which only are shown in Figure 5; the shapes marked 1, 2, 3, 6 have

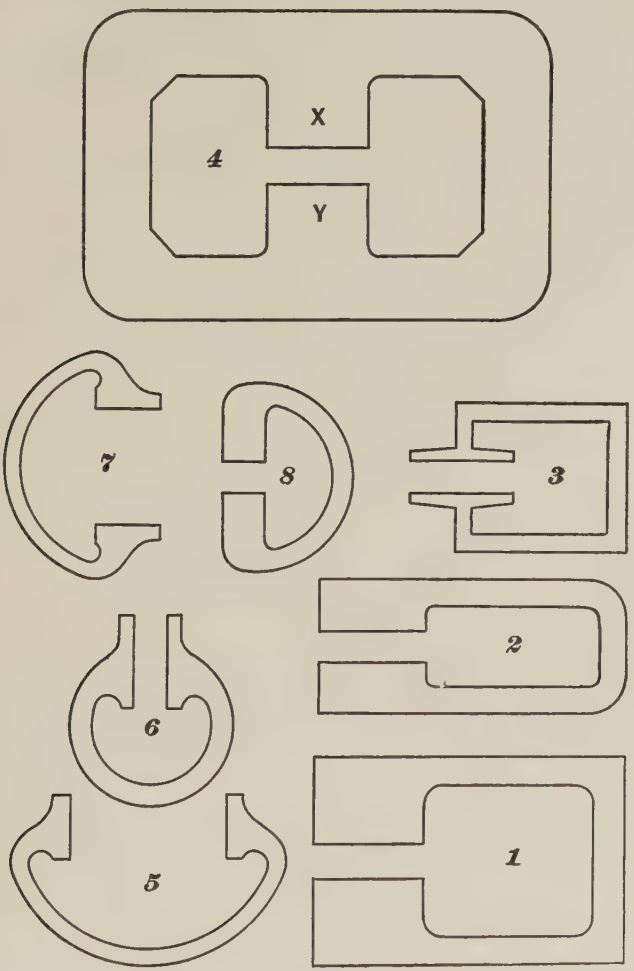


FIGURE 5

been employed, with the long way of the opening between the poles vertical, in d'Arsonval galvanometers, while a number of rather thin plates of the shape marked 8 have been used together in other instru-

ments of the same kind. Magnets of the shapes marked 5 and 7 produce the artificial fields in some mirror needle galvanometers used as voltmeters.

The mean temperature coefficient (K) between 10° C. and 100° C. of hardened magnets of the shape 1, which weigh 1250 grams apiece, is about 0.00036; K is the mean loss per degree of the intensity of the magnetic field at any definite point between the poles — when the magnet is heated from 10° to 100° — expressed in terms of the field intensity at the point at the lower temperature. For magnets of the shapes 3 and 6 — which weigh 260 grams and 500 grams respectively

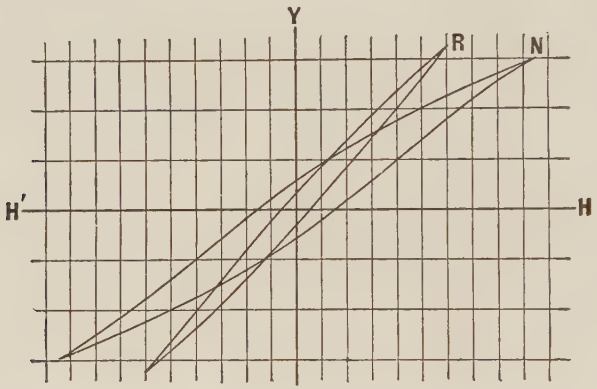


FIGURE 6

— K has the values 0.00040 and 0.00031. The actual temperature coefficients at room temperature are always less than these mean values, and in the case of the last-mentioned form the coefficient between 10° C. and 40° C. is not greater than 0.00013.

If the slender part of such a casting as No. 1 be wound as uniformly as possible with insulated wire, and if steady currents of different strengths be sent through this wire, the relative values of the whole magnetic induction across a given area between the poles can be determined with sufficient accuracy by pulling out, from a definite position between the jaws, a coil of suitable shape made of very fine manganine wire and connected with a ballistic galvanometer.

In this way diagrams may be obtained very like the C- and D-curves of Figure 3, according as the casting is soft or hard, and this is a very delicate method of determining whether a casting which is

hard on the surface is also hard throughout the interior. The two halves of each of two thick castings of the shape 4, weighing about 4800 grams each, were wound as uniformly as we well could with 156 turns of insulated wire, and the two coils of each casting were so connected in series that when a current was sent through them both conspired to make one of the projections (say X) a north pole and the other (Y) a south pole: one of the castings was soft while the other, as another experiment afterwards proved, had been imperfectly hard-

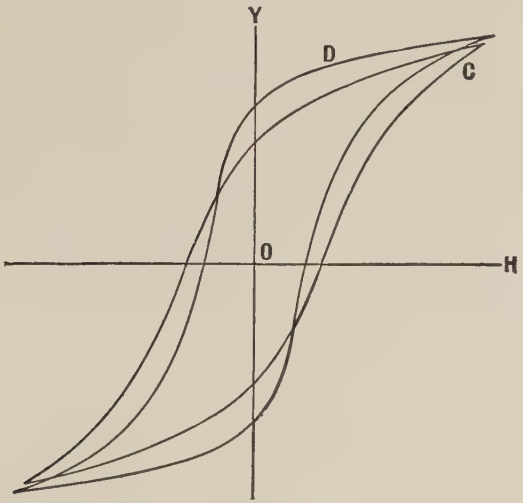


FIGURE 7

ened. With each of these castings a rough kind of hysteresis diagram was obtained by measuring for different current strengths the induction flux in the air between the poles; the ordinates in Figure 6 represent, in hundreds of lines per square centimetre, the induction across the centre of the gap X Y, while the abscissas represent the current in the coils measured in amperes; the R-curve belongs to the soft and the N-curve to the supposed hard casting. In a second experiment made with the same castings, one of the coils on each was used as a primary and the other, which was connected with a ballistic galvanometer, as a secondary; this procedure gave the curves of Figure 7, which show clearly that the second casting, which was very hard to the file on the surface, was still soft inside. The diagrams of

Figure 8, drawn on a different scale, give the results of a similar experiment on the soft casting just mentioned and a third chilled casting of the same form. These striking curves, which were reduced from a large drawing, represent the observations accurately, and illustrate

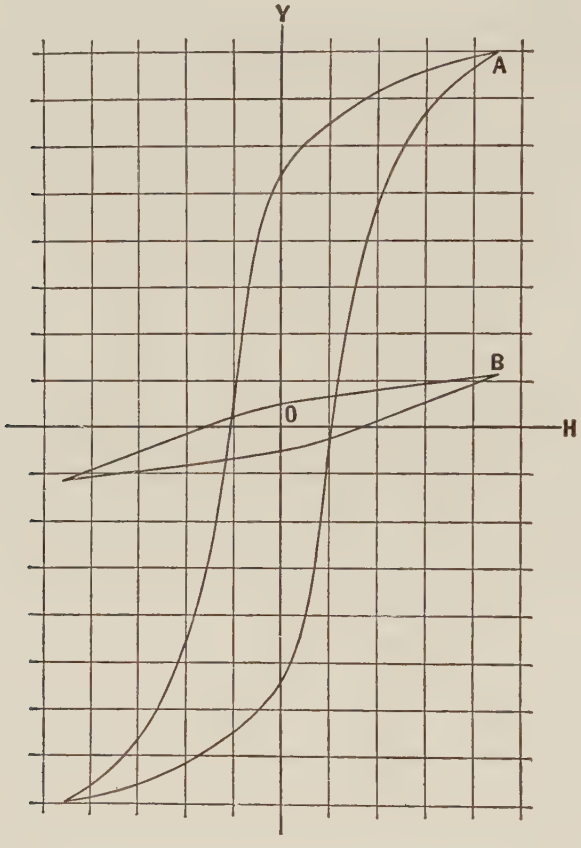


FIGURE 8

the fact that it is possible to make the whole inside of a massive casting, like the one here used, magnetically very hard.

When the gap X Y of this third casting had been closed by a piece of soft iron, and a heavy current had been sent through the coils for a few moments, in such a way as to make one of the projections a north pole and the other a south pole, the casting became a fairly strong perma-

nent magnet: the flux per square centimeter across the middle of the air gap was finally 285. The flux per square centimeter across the gap of a similar, though considerably less heavy, hardened cast-iron magnet, belonging to one of a number of excellent d'Arsonval galvanometers furnished by Messrs. Leeds and Northrup, is 187. In this magnet the pole projections are a little shorter and the gap a trifle wider than in the castings 4 described above.

V

ON THE MANNER OF GROWTH OF A CURRENT IN THE COIL OF A NEARLY-CLOSED ELECTROMAGNET AS INFLUENCED BY THE WIDTH OF THE AIR GAP [1]

In the course of a set of experiments on the time-lag in the magnetization of iron, which are being carried on in the Jefferson Laboratory, it has been found desirable to use, in series with the testing apparatus, one or more large electromagnets to increase the inductance of the whole circuit so greatly that the effects of sudden small changes in the inductance of the testing coils may be of slight importance; and it has been necessary to study specially some of the properties of these particular magnets, since, in their cases, certain of the conditions which underlie the convenient methods usually employed in treating practical problems are not very exactly fulfilled. In order to be able to predict the behavior of a nearly-closed magnet under the conditions of the work, one needed to know the effective inductance of the magnet under given conditions, and the manner of growth of the current in the coil circuit as dependent upon the applied electromotive force, the final value of the current, and the width of the air gap. These matters, with some others, were studied, and, apropos of the extremely interesting experiments recorded in Dr. Thornton's recent article on "The Magnetization of Iron in Bulk," [2] and of the work of Messrs. J. and B. Hopkinson and E. Wilson on "The Propagation of Magnetization of Iron as Affected by the Electric Currents in the Iron," [3] I propose to give briefly in this paper the results of tests made on one of the electromagnets I have used, as illustrating some characteristics of nearly-closed, massive iron cores. With these results I wish to compare some others obtained from a large closed electromagnet with finely laminated core.

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xli, no. 24, February, 1906.

[2] *The Philosophical Magazine*, vol. viii, 1904.

[3] *The Philosophical Transactions*, vol. clxxxvi, 1895. *Proceedings of the London Institute of Electrical Engineers*, 1895.

The general shape of the magnet in question is shown in Figure 1. The outside dimensions of the frame proper are about 101 cm. × 80 cm. × 40 cm. The base is of cast iron and of rectangular cross-section (20 cm. × 40 cm.), the cylindrical arms are of soft wrought iron 25 cm. in diameter, the rectangular pole pieces are 4.5 cm. thick, and the area of each of the opposed faces is about 580 square centimeters. The four coils, which were commonly used in series, have together 2823 turns and a resistance at 20° C. of about 12.4 ohms; the magnet weighs about 1500 kilograms.

The electromagnetic induction within so large a solid core as that of this magnet practically attains its final value, as is well known, only

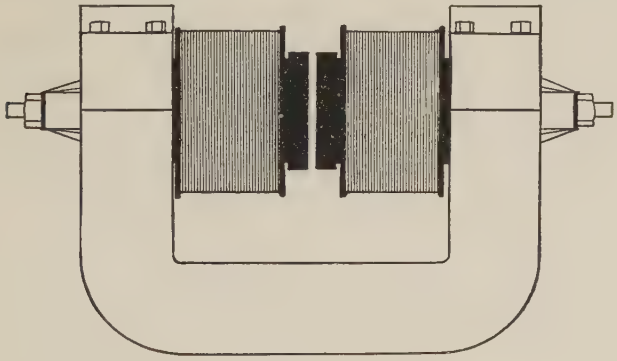


FIGURE 1

after an appreciable length of time. This time, for a given value of the electromotive force in the coil circuit, depends upon the amount of non-inductive resistance in the circuit outside the magnet, and, for a given value of the final exciting current in the coil, depends upon the applied electromotive force; under favorable circumstances it may be 200 seconds. At the outset Mr. J. Coulson and I obtained a large number of hysteresis curves for the core under given conditions, but for steady currents in the coil. The curves in Figure 2 are four of a series for different widths of the air gap, each obtained after the magnet had been put a number of times in succession through a Ewing's cycle with 6 amperes as the maximum current. The ordinates show the flux density at the centre of the gap in thousands of units, and the abscissas the current in amperes for the positive descending quarter

of the cycle; the rest of the figures are omitted to avoid confusion. The flux was measured by pulling a small thin coil of known dimensions, attached to a calibrated ballistic galvanometer, out of a pocket at the middle of the gap. The cylindrical arms of the magnet were held firmly by massive yokes outside the frame, but it was necessary to insert strips of non-magnetic material in the gap to prevent it from gradually closing by the bending of the frame when strong currents

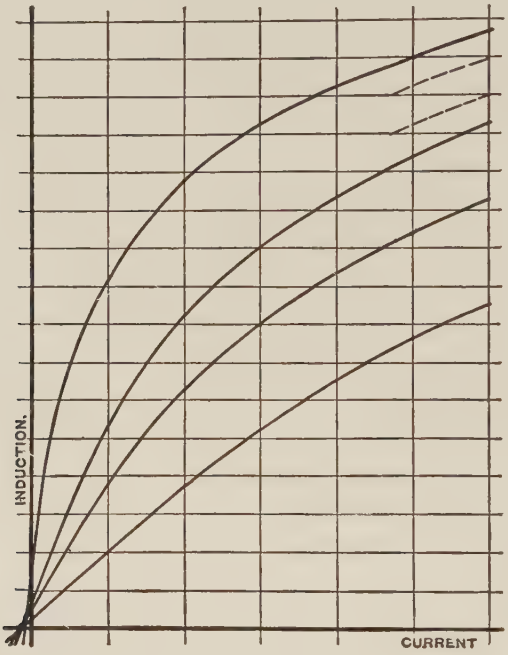


FIGURE 2

were used. These "chocks" were usually so inserted, by aid of gauges made for the purpose, that just one half of the gap — divided by a vertical line from the other half — was free. Counting from the top of the diagram, the full curves correspond to gap-widths of 1.6 mm., 6.6 mm., 9.8 mm., and 19.7 mm. respectively; between the first two others are bits of two curves belonging to gap-widths of 3.2 mm. and 4.7 mm.

The length of the line of induction which goes through the center of the pole pieces is about 250 cm., and it would be easy to find the form

of a curve, similar to those of Figure 2, for a closed gap by shearing [1] the upper curve of the diagram in the usual manner. When the gap was closed, a number of turns of insulated wire were wound directly about the core, and the ends of this coil were connected through an

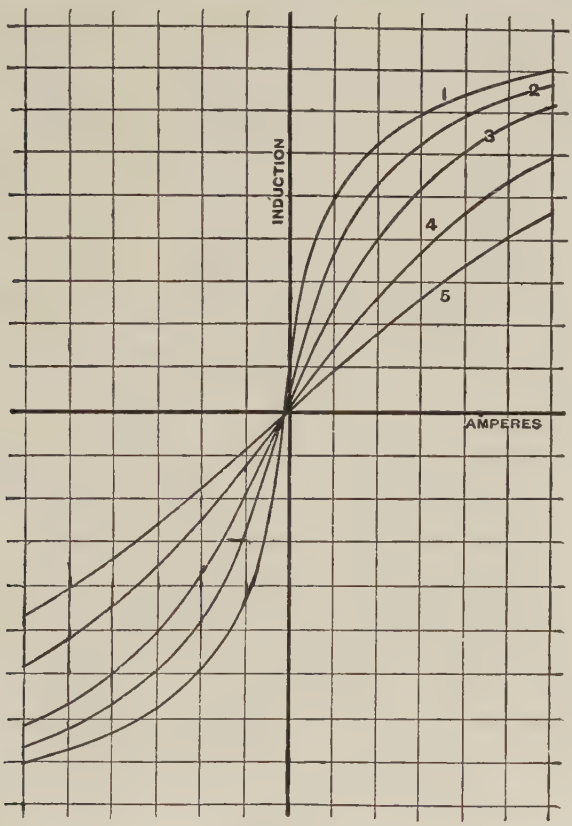


FIGURE 3

oscillograph which made its records on the same piece of paper which recorded the indications of another oscillograph in the main circuit. The main circuit contained also a massive rheostat of 200 ohms, total resistance, and the current in the circuit was made to grow from zero

[1] J. Hopkinson, *The Philosophical Transactions*, 1885; J. and B. Hopkinson, *The Philosophical Transactions*, 1886; B. Hopkinson, *Report of the Brit. Assoc. Adv. Sci.*, 1887; Ewing, *Magnetic Induction in Iron and Other Metals*, ch. 10; Du Bois, *Philosophical Magazine*, 1890.

by steps so far apart in time that, in every interval, the current in the secondary circuit had time to die sensibly out. After a maximum current of the desired intensity had been attained, the current was then reduced by steps to zero, and was then built up in the opposite direc-

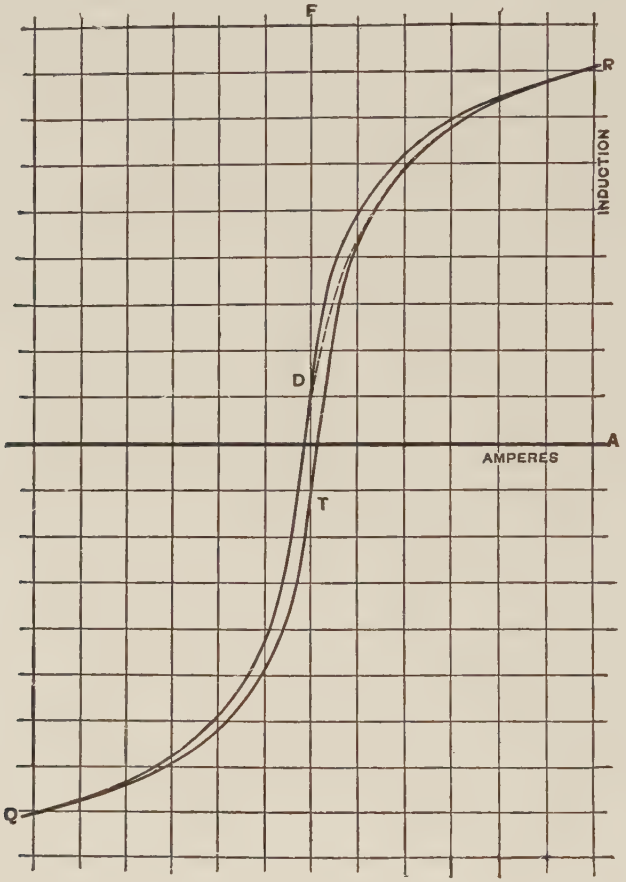


FIGURE 4

tion to the same maximum value by steps. The areas under the oscillograph curves of the secondary circuit evidently furnish means of obtaining a hysteresis curve for the core of the magnet, but this well-known method of procedure is, in my hands, and for this particular magnet, not quite so satisfactory as the one indicated above.

The leakage in a magnet of the form of this one is of course considerable, even when the gap is closed, and, for a given width of gap, the ratio of the induction through a circumference of, say, 48 cm. diameter in the plane of the gap with its centre (O) at the gap-centre, to the flux density at O, depends slightly upon the intensity of magnetization of the iron. With a gap-width of 28.3 mm. this ratio increased by about 3 per cent as the strength of the current in the coils rose from a small value to 6 amperes, and was sensibly the same for ascending and descending branches of a hysteresis cycle.

The ordinates of each of the curves in Figure 3 show, in millions of units, the mean total flux of induction, through each turn of the coils of the magnet, for a given gap-width, corresponding to currents represented in amperes by the abscissas. These curves are specimens of a set obtained experimentally: only half of each cycle — representing about 20 determined points, every one of which lies sensibly on the curve — is given, lest the complete diagrams prove confusing. Reckoned from the top of the diagram, the curves correspond, respectively, to the gap-widths 1.6 mm., 4.7 mm., 9.7 mm., 19.7 mm., and 28.4 mm. Figure 4 shows a similar cycle corresponding to a gap-width of about 1.6 mm. for a maximum current of about 6.2 amperes: the magnet was put repeatedly through the cycle before the observations were made. The dotted curve *DP* shows a rising branch of the cycle from *D* to *P*.

It is evident that the manner of growth of a current in the coil of the magnet is influenced by eddy currents in the core, by the residual effects of past magnetic experiences and the corresponding form of a hysteresis cycle for rapid changes of the magnetizing field, and by magnetic lag, if such there be, as well as by the causes enumerated above. It is always difficult to distinguish between the effects of all these causes, and it will be well to get such help as we can from a theoretical discussion of the effect of eddy currents alone in a core of definite constant permeability.

Eddy Currents in a Core of Fixed Permeability within a Long Solenoid

Several writers have discussed the application of Maxwell's general equations [1] to the determination of the growth of currents in coils of wire which surround solid metal cores of various forms, and Heaviside printed more than twenty years ago [2] an extremely interesting series of fifteen papers on problems connected with the induction of currents in the solid core of a long solenoid. Although it is practically impossible to subject to accurate computation the growth and the decay, under given conditions, of currents in the coils of a magnet like that shown in Figure 1, it will be instructive to consider some analogous problems in the case of a long solenoid, the solid core of which is supposed to have a fixed permeability, and to be of the same diameter as that of the iron cylinders within the coils of the magnet in question. To facilitate comparison between the numerical results of this paper and those obtained in similar cases by Heaviside, it will be convenient to use his notation, at least in part.

A long, solid, circular, iron cylinder, of specific resistance ρ and of radius a, is closely surrounded by a uniformly wound coil of wire which has N turns per centimeter of the length of the core; the outer radius of the coil is $(a + b)$ and the axis of the core is the z axis. A current (C) in the coil is accompanied by a magnetic field (H) in the core which has the direction of the z axis, and any change in the intensity of C induces in the core temporary currents, the lines of which are circles parallel to the xy plane with centers on the z axis. At any instant the value of H and that of q, the vector which gives the density of the current at any point in the core, are functions of the distance (r) from the z axis alone, and are independent of z; hence Maxwell's current equation,

$$4\pi q = \text{Curl } H, \tag{1}$$

reduces to the simple form

$$4\pi q = -\frac{\partial H}{\partial r}. \tag{2}$$

[1] *Treatise on Electricity and Magnetism*, vol. ii, ch. 9.
[2] *The Electrician*, 1884–85; *Electrical Papers*, i, 353–416.

The currents in the core do not affect the intensity (H_a) of the magnetic field at the boundary of the coil, so that at every instant

$$H_a = 4\pi NC. \tag{3}$$

Since in columnar co-ordinates

$$\nabla^2(V) \equiv \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial V}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}, \tag{4}$$

where V is any scalar function, the general equation,

$$\frac{4\pi\mu}{\rho} \cdot \frac{\partial H}{\partial t} = \nabla^2(H), \tag{5}$$

becomes

$$\frac{4\pi\mu}{\rho} \cdot \frac{\partial H}{\partial t} = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial H}{\partial r}\right), \tag{6}$$

where μ is the permeability of the iron, supposed constant.

When there are no currents in the core, the intensity (H) of the magnetic field in the core has at every point the boundary value (H_a), but when positively directed eddy currents exist, the intensity of the field is greater near the axis and sinks gradually, as r increases, to H_a at the boundary. If, then, L is the inductance of the coil per centimeter of the length of the core, when the core is without currents and there are no other circuits in the neighborhood so that the magnetic field within the coil is uniform, this inductance will be increased by an amount L' when, in consequence of Foucault currents running right-handedly around the axis, the intensity of the field within the core is raised above $4\pi NC$. The contribution L' comes from a field every line of which threads every turn of the coil. We have, therefore,

$$L' = N\int_0^a \mu(H - H_a)\,2\pi r dr = 2\pi\mu N\int_0^a Hr dr - 4\pi^2 a^2 N^2 \mu C, \tag{7}$$

and if the coefficient of C in the last term be denoted by L_1, the whole induction flux through the turns of the coil per centimeter of the length of the solenoid is

$$p \equiv (L - L_1)\,C + 2\pi\mu N\int_0^a Hr \cdot dr; \tag{8}$$

if w is the uniform resistance of the coil per centimeter of the length

of the core and E the impressed electromotive force in the coil circuit, reckoned in the same way,

$$E - \frac{dp}{dt} = wC, \tag{9}$$

or
$$E = wC + (L - L_1)\frac{dC}{dt} + 2\pi\mu N \int_0^a \frac{\partial H}{\partial t} \cdot r \cdot dr; \tag{10}$$

or, by virtue of (6),

$$E = wC + (L - L_1)\frac{dC}{dt} + \tfrac{1}{2} N\rho a \left(\frac{\partial H}{\partial r}\right)_{r=a}. \tag{11}$$

To fix one's ideas, one might imagine every centimeter of the length of the coil (measured parallel to the core axis) to be a separate circuit containing an applied electromotive force of E absolute units (the same for every such circuit) and having a total resistance w made up of the resistance (w') of the wire actually wound on the core, and the resistance (w'') of the external part of the circuit which is non-inductive.

If, in order to determine a set of normal special solutions of the linear equation (6), we assume H to be the product of a function (T) of t alone, and a function (R) of r alone, and substitute this product in the equation, we arrive at the well-known normal form

$$e^{-a^2t}[A \cdot J_0(nr) + B \cdot K_0(nr)], \tag{12}$$

where either a or n may be assumed at pleasure and the other computed by means of the equation

$$\rho n^2 = 4\pi\mu a^2. \tag{13}$$

Since the core of the solenoid is solid, Bessel's Functions of the second kind will not be needed in the problems of this paper, and we may assume that H is expressible in an infinite series of terms of the form

$$A \cdot e^{-a^2t} \cdot J_0(nr). \tag{14}$$

If, after the current in the coil has been for some time steady and the core has become uniformly magnetized, the coil circuit be suddenly broken so that the duration of the spark is less than a thousandth of a second, the intensity of the magnetic field at the boundary of the core where $r = a$ falls suddenly to, and remains thereafter at,

zero, and the normal form (14) will satisfy the condition $H_a = 0$, if such a value be chosen for n as shall make na a root of the equation

$$J_0(x) = 0. \tag{15}$$

A sufficient number of these roots for the purposes of this paper can be found [1] in almost any book on Bessel's Functions, with the corresponding values of $J_1(x)$: the first twelve are given in Table I.

The pth root in order of magnitude of the equation $J_1(x) = 0$ is denoted by x_p.

TABLE I

p	x_p	$J_1(x_p)$	p	x_p	$J_1(x_p)$
1	2.404826	+0.519148	7	21.211637	+0.173266
2	5.520078	−0.340265	8	24.352472	−0.161702
3	8.653728	+0.271452	9	27.493479	+0.152181
4	11.791534	−0.232460	10	30.634606	−0.144166
5	14.930918	+0.206546	11	33.775820	+0.137297
6	18.071064	−0.187729	12	36.917098	−0.131325

If the intensity of the uniform magnetic field in the core before the break was H_0, we have [2] at any time (t) after the break and at any distance (r) from the axis of the core,

$$H = \frac{2H_0}{a} \sum_p \frac{J_0(n_p r)}{n_p \cdot J_1(n_p a)} e^{-\alpha_p^2 t}, \tag{16}$$

and, since

$$\frac{dJ_0(nr)}{dr} = -n \cdot J_1(nr),$$

$$q = \frac{+H_0^9}{2\pi a} \sum_p \frac{J_1(n_p r)}{J_1(n_p a)} e^{-\alpha_p^2 t}. \tag{17}$$

In the case here treated the diameter of the core is 25 centimeters ($a = 12.5$), and for the kind of iron used, at room temperatures, we may write

$$\alpha^2 = \frac{5.12(na)^2}{\mu}. \tag{18}$$

The negative of the time rate of change of the total flux of induction through a cylindrical surface of radius r coaxial with the core and

[1] Byerly, *Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics*, p. 286; Gray and Mathews, *Treatise on Bessel's Functions*, p. 244; Peirce and Willson, "The first 40 roots of the Equation $J_0(x) = 0$," *Bulletin of the American Mathematical Society*, 1897.

[2] Byerly, *Treatise on Fourier's Series*, etc., p. 229; Heaviside, *Electrical Papers*, i, 391.

lying within it, is at every instant proportional to the expression for q given above. The values of $J_0(x)$ and $J_1(x)$ for every hundredth of a unit between $x = 0$ and $x = 15.50$ are given in Meissel's Tables [1] to twelve decimal places, and after the proper value of μ has been introduced into (18) and the value for r chosen, it is not very difficult, ex-

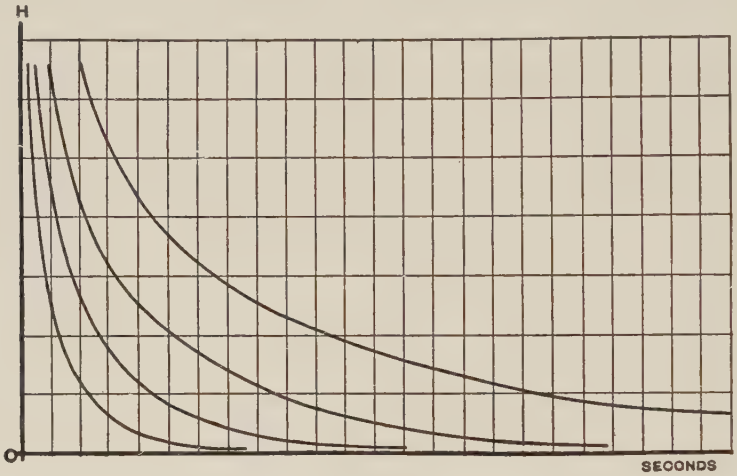


FIGURE 5

cept in the case of small values of t, to compute the value of the series (S) in (17) for different epochs. If, for example, μ is 40, and if we consider a point at the boundary of the core, S has the values given in the next table. The time is of course measured in seconds.

TABLE II

t	S	t	S
0.25	1.3120	3.00	0.1085
0.50	0.8413	4.00	0.0518
0.75	0.6285	5.00	0.0247
1.00	0.4974	6.00	0.0118
2.00	0.2227	8.00	0.0026

Figure 5 shows four curves in which S is plotted against t for $r = a$ and $\mu = 20, 40, 80$, and 160 respectively. If the circuit of a few turns of fine insulated wire wound directly on the core were closed through an oscillograph, and if μ were independent of H and there were no

[1] Meissel, *Tafel der Bessel'chen Functionen*, Berliner Abhandlungen, 1888; Gray and Mathews, *Treatise on Bessel's Functions*, pp. 247–266.

time-lag in its magnetization, the records should show curves like these. For large values of μ the current would decay very slowly. The actual values of the ordinates of any curve would depend of course upon the number of turns in the secondary, the resistance of its circuit, and the magnetic constants of the solenoid. If for a fixed point in the core a curve be drawn, by plotting q against t, for each of a number of different values of μ, the ordinates of all these curves will have equal values at points where t/μ has the same value. The maximum value of q, if it has one, is independent of μ.

At a point distant one tenth of the radius of the core from the axis

$$q = \frac{H_0}{2\pi a} \sum \frac{J_1(\frac{1}{10} n_p a)}{J_1(n_p a)} e^{-a p^2 t} = \frac{H_0 S'}{2\pi a}, \tag{19}$$

and ignoring algebraic signs, we have

TABLE III

p	$\log \frac{J_1(\frac{1}{10} n_p a)}{J_1 n_p a}$	p	$\log \frac{J_1(\frac{1}{10} n_p a)}{J_1 n_p a}$
1	9.3617	7	0.5141
2	9.8924	8	0.5009
3	0.1611	9	0.4473
4	0.3264	10	0.3395
5	0.4308	11	0.1378
6	0.4911	12	9.6398

Table IV gives to four decimal places the values of S', for $\mu = 40$ and $r = 1.25$, at a number of different epochs after the coil circuit has been broken.

TABLE IV

t	S'	t	S'
0.17	0.0004	1.50	0.0736
0.25	0.0062	2.00	0.0520
0.50	0.0597	2.50	0.0361
0.60	0.0770	3.00	0.0250
0.75	0.0912	4.00	0.0119
0.90	0.0948	5.00	0.0057
1.00	0.0940	6.00	0.0027
1.25	0.0853	8.00	0.0008

If a small closed testing coil of fine insulated wire could be so imbedded in the iron as to surround a portion of the core coaxial with the whole and of one tenth the whole diameter, the intensities of the current in the coil at different times should be proportional to S'. Figure 6

shows four curves [1] in which S' is plotted against t for $\mu = 20, 40, 80,$ and 160.

The maximum value of q at $r = a/10$ may be found by equating to zero $\partial q/\partial t$ obtained from (19)

$$\frac{\partial q}{\partial t} = -\frac{2.56\,H_0}{\pi\mu a}\sum\frac{(n_p a)^2 J_1(\frac{1}{10}n_p a)\cdot e^{-a_p{}^2 t}}{J(n_p a)}. \qquad (20)$$

If for $r = a/10$, q be plotted against t for a number of values of μ, t/μ will have the same value at the highest point of all the curves.

In order to determine the manner of growth of the current in the coil of the solenoid when the circuit is suddenly closed, it will be well to

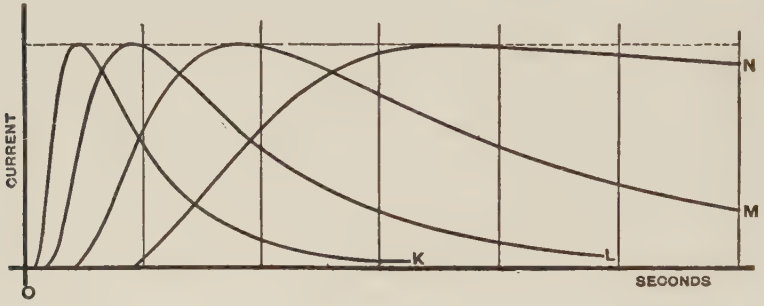


FIGURE 6

follow the usual procedure in treating analogous problems in heat conduction, and inquire first how the coil current would decay if, after the core has been uniformly magnetized by a steady coil current, the electromotive force were suddenly cut out of the coil without opening the circuit. The solution of this problem furnishes immediately the solution of the one first stated. Heaviside [2] has treated by this method a solenoid with iron core 2 centimeters in diameter. If the coil is fairly thin, so that substantially the whole flux of induction which threads its turns is the flux in the core itself, we need not distinguish between L and L_1 in equation (11) and since there is no electromotive force in the coil

$$C + \frac{N\rho a}{2w}\left[\frac{\partial H}{\partial r}\right]_{r=a} = 0, \qquad (21)$$

[1] Each curve practically coincides with the axis of abscissas for a time, and then suddenly bends sharply away from the axis. It is not easy to indicate these sharp bends in a small figure.

[2] Heaviside, *Electrical Papers*, i, 394.

and by virtue of (3)

$$H_a + \frac{4\pi N^2 \rho a}{2w}\left[\frac{\partial H}{\partial r}\right]_{r=a} = 0, \tag{22}$$

or

$$H_a + s \cdot \left[\frac{\partial H}{\partial r}\right]_{r=a} = 0. \tag{23}$$

The constant s is to be determined from the constants of the solenoid and of the core, and we shall find it instructive to study the effect of a change in s in a problem otherwise given. We will assume at first that N, the number of turns of wire in the coil per centimeter of the length of the core, and w, the resistance in absolute units of the coil per centimeter of its length, are such as to make s unity; in the second case the value of s shall be 2.5. The first is somewhat less, the second much greater than the value which would most closely correspond to the magnet shown in Figure 1. The field intensity, H, in the core must satisfy equations (6) and (23) at every instant, and, when $t = 0$, must be equal to H_0 for all values of r.

The special solution,

$$Ae^{-a^2 t} \cdot J_0(nr), \tag{24}$$

of (6), in which

$$a^2 = \frac{\rho (na)^2}{4\pi \mu a^2},$$

satisfies (23) provided that na is a root of the equation

$$J_0(na) = \frac{nas}{a} J_1(na), \tag{25}$$

and if we use the successive values [1] of n

$$1 = \sum \frac{2 \cdot J_0(n_p r)}{n_p a (1 + s^2 n_p^2) J_1(n_p a)}, \tag{26}$$

so that

$$H = \frac{2H_0}{a} \sum \frac{e^{-a_p^2 t} \cdot J_0(n_p r)}{n_p (1 + s^2 n_p^2) J_1(n_p a)}, \tag{27}$$

and

$$H_a = \frac{2H_0 s}{a} \sum \frac{e^{-a_p^2 t}}{1 + s^2 n_p^2}. \tag{28}$$

In the case here treated na is a root of the equation

$$J_0(z) = \frac{2z}{25} J_1(z), \tag{29}$$

[1] Byerly, *Treatise on Fourier's Series*, etc., p. 229.

and it is not difficult to prove by the aid of Meissel's Tables that the first five roots have approximately the values given below.

TABLE V

$$\begin{aligned} n_1 a &= 2.2218 \\ n_2 a &= 5.1171 \\ n_3 a &= 8.0624 \\ n_4 a &= 11.0476 \\ n_5 a &= 14.0666 \end{aligned}$$

Since
$$4\pi q = -\frac{\partial H}{\partial r} \tag{30}$$

and
$$\frac{dJ_0(nr)}{dr} = -n \cdot J_1(nr), \tag{31}$$

$$q = \frac{H_0}{2\pi a} \sum \frac{e^{-\alpha_p^2 t} J_1(nr)}{(1 + s^2 n^2) J_1(na)}. \tag{32}$$

The function defined by (27) satisfies (23) when $r = a$ for all values of t, and equation (6) for all points within the core for all positive values of t; when $t = 0$, the function is in the core everywhere equal to H_0, and when t is infinite, it vanishes. It is easy to see, therefore, that a function H defined by the equation

$$H \equiv H_\infty - \frac{2H_\infty}{a} \sum \frac{e^{-\alpha_p^2 t} \cdot J_0(n_p r)}{n_p(1 + s^2 n_p^2) \cdot J_1(n_p a)}, \tag{33}$$

where H_∞ is a given constant, satisfies equations

$$H_a + s[D_r H]_{r=a} = H_\infty$$

and (6), is everywhere equal to zero when $t = 0$, and, when t is infinite, is everywhere equal to H_∞. Since a function which satisfies these conditions is unique, (33) represents the strength of the magnetic field within the core of the solenoid while it is being magnetized from a neutral state to the uniform intensity H_∞ by a current (C) in the coil due to an electromotive force impressed in it. If N is the number of turns of wire in the coil, E the applied electromotive force in the coil circuit, and $w = w' + w''$ the resistance of the coil circuit, all three per centimeter of length of the solenoid, the coil current is given (11) by the equation

$$C = \frac{E}{w} - \frac{\frac{1}{2}N\rho a}{w}\left[\frac{2H_\infty}{a} \sum \frac{e^{-\alpha_p^2 t}}{1 + s^2 n_p^2}\right]. \tag{34}$$

It is possible to hasten the growth of a current of given final value in a simple circuit with fixed inductance (L) independent of the current strength, by increasing the applied electromotive force (E), and adding to the circuit a corresponding amount of resistance wound non-inductively; for this process decreases the time-constant L/r without changing E/r. The same statement is true in practice for almost every sort of electromagnet.[1] It is not easy to see immediately

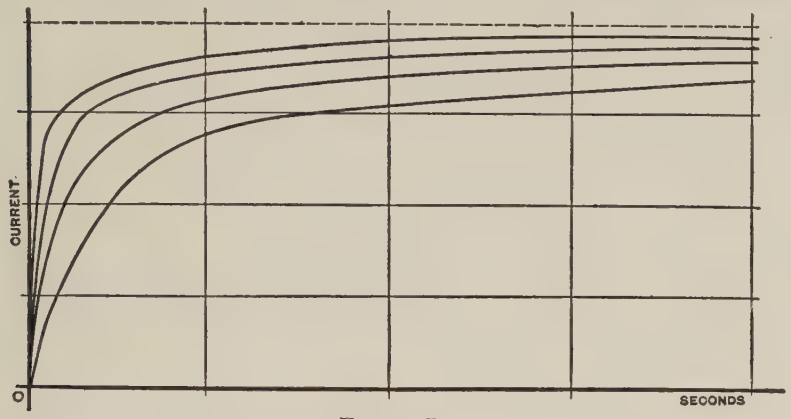


FIGURE 7

from equation (34), however, just what the effect on C is of a given change in w, for both n and a involve w implicitly through s.

Since $H_\infty = 4\pi NC_\infty$, we may rewrite (34) in the form

$$C = C_\infty \left[1 - \frac{2s}{a} \sum \frac{e^{-a_p{}^2 t}}{1 + s^2 n_p{}^2}\right]. \tag{35}$$

For $\mu = 40$, and $s = 1$, the series and the parentheses which appear in this equation have at different times the approximate values given in Table VI.

Figure 7 shows C plotted against t for $\mu = 40$, 80, 160, and 320.

Since $4\pi q = -\partial H/\partial r$, it follows from equation (33) that the value of q in the case of the growing C is the negative of the value given by equation (32) for the case of decaying C. Figure 8 shows the value of q at different times for $\mu = 40$, 80, and 160, $r = a/10$, and $s = 1$.

[1] After this paper was in type I became acquainted with the results of the elaborate study made by Professor T. Gray into the manner of growth of currents in the coils of electromagnets with finely divided cores. The beautiful curves which he gives in vol. clxxxiv of the *Philosophical Transactions* of the Royal Society illustrate very strikingly the fact here mentioned.

TABLE VI

t	Σ	$\frac{2s}{a}\Sigma$	$\left[1-\frac{2s}{a}\Sigma\right]$
0.00	6.2500	1.0000	0.0000
0.25	1.2985	0.2078	0.7922
0.50	0.8783	0.1405	0.8595
0.75	0.6743	0.1079	0.8921
1.00	0.5454	0.0873	0.9127
1.25	0.4530	0.0725	0.9275
1.50	0.3814	0.0610	0.9390
2.00	0.2748	0.0440	0.9560
3.00	0.1456	0.0233	0.9767
4.00	0.0775	0.0124	0.9876
6.00	0.0219	0.0035	0.9965
8.00	0.0062	0.0010	0.9990
10.00	0.0017	0.0003	0.9997

When $s = 2.5$, the equation corresponding to (29) is

$$5 \cdot J_0(z) = z \cdot J_1(z), \tag{36}$$

and the first five roots have approximately the values given below.

TABLE VII

$n_1 a = 1.9898$
$n_2 a = 4.7132$
$n_3 a = 7.6177$
$n_4 a = 10.6223$
$n_5 a = 13.6786$

and for $\mu = 40$, the series of (35) has approximately the values given in Table VIII.

TABLE VIII

t	Σ	$\frac{2s}{a}\Sigma$	$\left[1-\frac{2s}{a}\Sigma\right]$
0.00	2.5000	1.0000	0.0000
0.25	1.0729	0.4292	0.5708
0.50	0.8052	0.3221	0.6779
0.75	0.6542	0.2617	0.7383
1.00	0.5511	0.2204	0.7796
1.25	0.4732	0.1893	0.8107
1.50	0.4111	0.1644	0.8356
2.00	0.3151	0.1260	0.8740
3.00	0.1889	0.0756	0.9244
4.00	0.1137	0.0455	0.9545
5.00	0.0685	0.0274	0.9726
6.00	0.0413	0.0165	0.9835
8.00	0.0150	0.0060	0.9940
10.00	0.0054	0.0022	0.9978

Suppose that in the case of the solenoid the resistance (w') of the coil alone, per centimeter of the length of the core, is such that when there is no outside resistance in the coil circuit, s is 2.5, and let us study the effect upon the manner of growth of C, of adding resistance wound non-inductively to the coil circuit to such an amount (w'') per unit of length of the core that s becomes 1.

From (35), we get

$$\frac{C_\infty - C_t}{C_\infty} = \frac{2s}{a} \sum \frac{e^{-q_p^2 t}}{1 + s^2 n_p^2}, \tag{37}$$

and Tables VI and VIII show that the second member of this equation is greater for every value of t after the beginning when s is 2.5 than when s is 1, whatever the value of μ may be. The first member de-

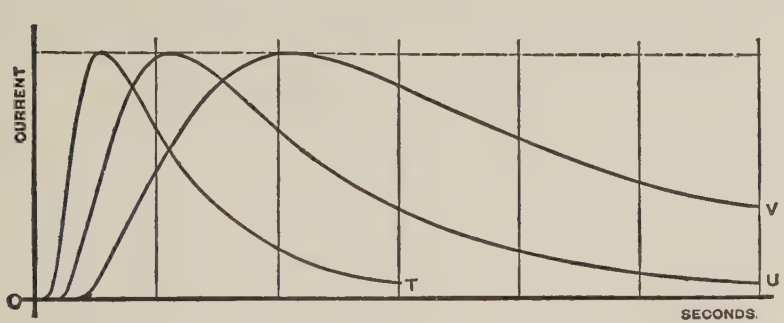


FIGURE 8

notes the fractional part of the final current which the actual current has at the time t still to attain; if, then, the intensity of the final current ($C_\infty = E/w$) be fixed, and if the current be built up in the coil circuit, first, when $w = w'$ and $s = 2.5$ and the value of E is correspondingly low; and second, when $w = \omega' + \omega''$, $s = 1$, and E has a correspondingly high value, the actual current will lag behind the final current by a smaller amount at every instant in the second case than in the first. Again, if E be fixed and if different values be given to w so that C_∞ has different values the actual current lags behind the final current by a smaller fraction of the latter at every instant when s is 1 than when $s = 2.5$, that is, when w is large than when w is small.

The quantity s, when the geometrical conditions are fixed, is in-

versely proportional to w, but is independent of μ, as is also n; a^2 is, however, inversely proportional to μ, and for two different values of μ, e^{-a^2t} would have the same numerical value at times which are to each other as these values of μ. It is sufficiently well proved that the effective permeability of the iron core of an electromagnet, when a current is rising rapidly in the coil, is not always the same as the permeability belonging to the instantaneous value of the current as determined from a statical hysteresis diagram. If in any case where w is fixed the effective value of μ should be greater or smaller for an increase in the value of the applied voltage E, the growth of the current would be relatively retarded or accelerated. We shall find it well to return to this subject later on.

The Effect of Variation in the Permeability of the Core of an Electromagnet upon the Manner of Growth of a Current in the Coil

We may add to the foregoing theoretical discussion of the effects of Foucault currents in a solid core the permeability of which has a fixed value, a few words upon the manner in which a current might be established in the coil of the magnet described in this paper if there were no eddy currents in the core, as, of course, there really would be in an actual case, unless the core were divided.

After the form of the dotted curve DP in Figure 4 had been determined with considerable accuracy (the observations of different days agreeing with each other almost exactly), the curve was plotted on a very large scale by means of a needle point upon thin sheet zinc and an accurate template was then cut out; with the help of this and a metal straight-edge, I measured as carefully as I well could the slope (λ) of this curve for a large number of different values of the current (i). Since 2.823×10^9 times an ordinate of the DP curve shows the total flux (F) through the coil for a current represented by the corresponding abscissa in amperes, then, if there were no eddy currents in the core and no time-lag in the magnetization of the iron, the building up of a current in the coil under the given circumstances on the application of a steady voltage E in a circuit of total resistance r ohms would be dominated by the equation

$$E - ri = 28.23\,\lambda \cdot \frac{di}{dt}, \tag{38}$$

(or)
$$\frac{dt}{di} = (28.23)\,\frac{\lambda}{E - ri}, \tag{39}$$

in which the second member is now a known function of i. I plotted this function and determined by aid of an Amsler's Planimeter the values of t for a number of values of i in the actual case, where E was 84.0 and $r = 13.55$. Figure 9 shows the building-up curve (Q) which

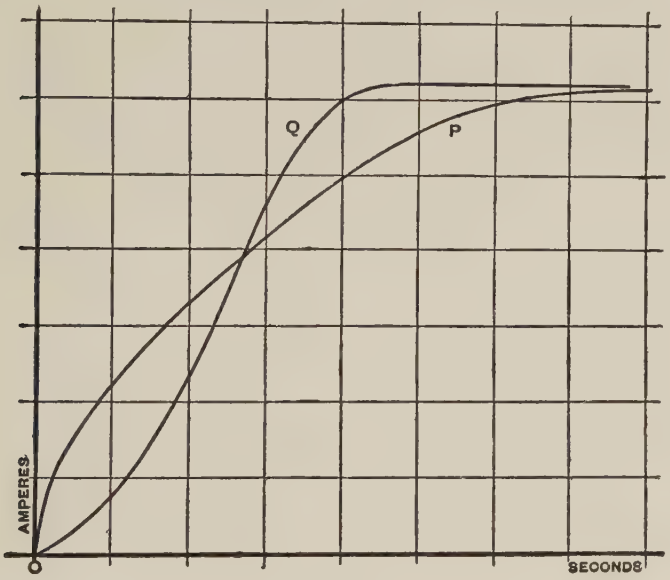


FIGURE 9

this process yields, and also the actual curve [1] carefully reproduced from an oscillograph record.

If we were to define the inductance of the magnet in a condition represented by a point R on the dotted line DP in the statical hysteresis diagram, as the ratio of the total flux which then passes through the coils to the intensity of the current belonging to the point, its value would decrease when the current increased, in the manner shown in the subjoined table.

[1] J. and B. Hopkinson, *The Electrician*, September, 1892; J. Hopkinson, Wilson and Lydall, *Proceedings Royal Society*, vol. liii.

TABLE IX

Current	"Inductance" in Henries	Current	"Inductance" in Henries
0.5	169.4	3.5	57.4
1.0	124.2	4.0	51.9
1.5	99.8	4.5	47.5
2.0	84.0	5.0	43.7
2.5	72.3	5.5	40.5
3.0	64.0	6.0	38.1

If, however, we write

$$\frac{d(Li)}{dt} = E - ri \tag{40}$$

and plot from the actual building-up curve given in Figure 9, $E - ri$ as a function of t, we may get the area under this curve by means of a planimeter for a series of values of t, and determine L from the equation

$$Li = \int_0^t (E - ri)\, dt. \tag{41}$$

The "apparent inductance" thus defined is affected by Foucault currents; its values for several points of the curve are given in Table X.

TABLE X

Time	Current	L	Time	Current	L
0.1	0.55	80.2	3.5	4.64	44.1
0.5	1.55	70.4	4.0	5.03	40.9
1.0	2.25	64.0	4.5	5.35	37.9
1.5	2.84	59.1	5.0	5.62	35.1
2.0	3.33	54.9	5.5	5.84	32.4
2.5	3.79	51.9	6.0	5.99	30.0
3.0	4.23	47.4			

L plotted against i yields a curve that is fairly straight.

The theoretical curve (Q) is very like some [1] of the actual curves which have been obtained from magnets with finely divided cores, whereas the actual building-up curve represented in the diagram is similar to those [2] which almost any electromagnet with solid core can be made to yield. The eddy currents, the changes in effective per-

[1] See, for instance, the fine diagram given by Dr. Thornton, *Phil. Mag.*, 1904, p. 625.

[2] J. Hopkinson and Wilson, *Phil. Trans.*, 1895, pp. 275, 280.

meability, and the other disturbing influences taken together, do not in such a case cause the curve to deviate very widely in shape from that which one could get from a simple circuit with fixed inductances and no eddy currents: the resemblance can almost always be made close by proper choice of the electromotive force of the exciting battery.

After I had determined hysteresis diagrams (some of which are shown in Figure 3) for the magnet for a large number of gap-widths up to 28 mm., it seemed likely that for an air gap about 35 mm. wide the hysteresis diagram would not be very different from a single straight line. For this gap-width the induction flux through the coil should be practically proportional to the strength of the current and amount to about 6.45×10^6 for a current of one absolute unit so that the inductance of the coil circuit should be about (28.23) (0.645) henries for a wide range of currents. The form of the building-up curve of a current in the coil of an electromagnet generally depends very much upon the magnetic state of the iron at the outset. If a steady current which has been running through the coil for some time be interrupted, and if then after a little the circuit be closed again, the manner of growth of the new current is generally very different if this current has the same direction as its predecessor or the opposite direction: that is, if the magnetism of the core follows the hysteresis diagram in a direction corresponding to *DP* in Figure 4 or in the direction *DQ*. An easy way of testing whether the hysteresis curve of a *large* electromagnet has an insignificant area is to obtain large oscillograph records of the building-up curves of direct and reversed currents and to compare the two. I took, therefore, a series of building-up curves for a gap-width of 35 mm., using currents of 2.75 amperes and 5.60 amperes, and found that in both cases the curves were wholly indistinguishable [1] even when enlarged and superposed on the screen, whether the current in question had the same direction as its predecessor or the opposite direction. With this gap-width, therefore, the magnet is an example of a circuit "containing iron" with an induction flux for *steady currents* almost exactly proportional to the strengths of these currents, and in this sense with a fixed inductance. If there were no eddy currents, and no time-lag in the magnetization of the core, the

[1] For a similar case, see Professor T. Gray, *Phil. Trans.*, vol. clxxxiv.

growth of the current in the coil should follow the law

$$C = \frac{E}{r}(1 - e^{-rt/L}),$$

where L is this fixed inductance. Figure 10 shows the actual oscillograph records for 2.75 amperes and 5.60 amperes in full line, and the theoretical curve in dotted line for $E = 80$. In this case, where the

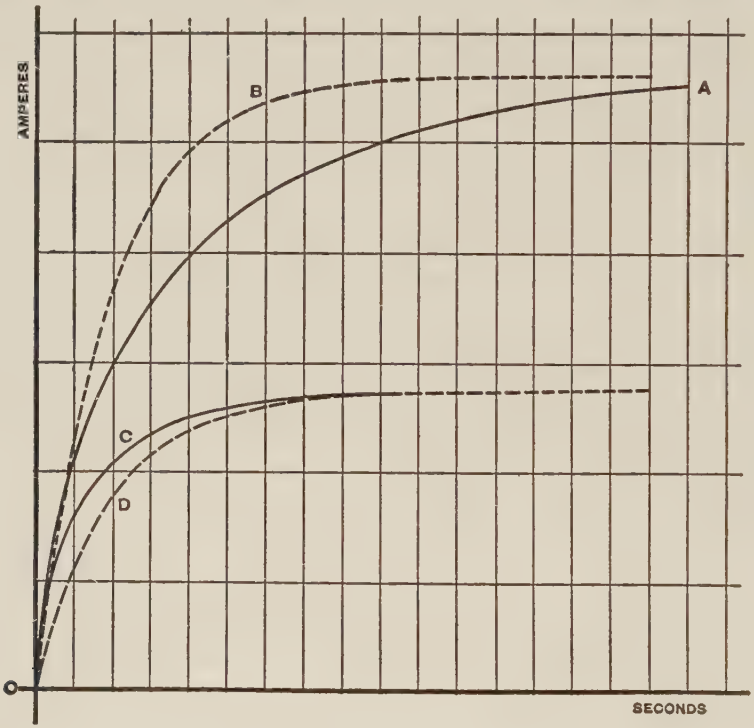


FIGURE 10

"statical effective" value of μ is independent of the current, but where eddy currents and what we may term time-lag in the taking up of the magnetism by the iron may enter, it is interesting to see that in spite of the retardation due to eddy currents, the current in the main circuit builds up more quickly than would correspond to the statical value of μ when the current is 2.75 amperes and that it starts to do so when the current is 5.6 amperes.

The building-up curves shown in this paper are careful reproductions of oscillograph records, of which I have several hundreds. Some of these were obtained with the aid of a Duddell Double Oscillograph, the drum of which could be turned either by an electric motor from the alternating street circuit or by clock-work, but most of them I got with the help of two single instruments made by Mr. J. Coulson, who helped me to take the photographs, and these served their purpose admirably. One of them, which was used in measuring comparatively small induction currents and needed to be very sensitive, was not

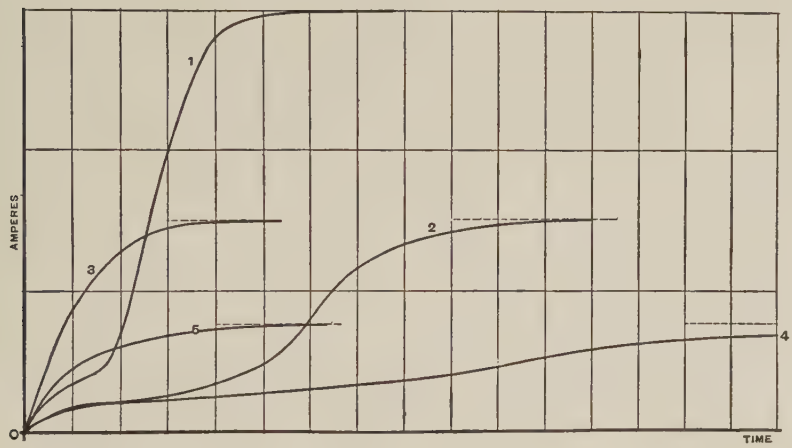


FIGURE 11

quite aperiodic when suddenly deflected to the end of its scale, but this fact did not affect records of the kind used here. This instrument consisted merely of a mirror galvanometer in which the extremely minute magnet and mirror were fastened to a piece of fine stretched gimp damped in oil. I had four drums for carrying the sensitized paper, or the film when this had to be used, and any one of these could be driven very uniformly at almost any rate up to a rim velocity of a meter per second by means of chronograph clock-work. Most of the paper used was in strips from 16 to 20 centimeters wide and some of the curves are about a meter long. The deflections of each oscillograph were strictly proportional to the current in its coil. A large zinc template, which carefully kept the irregularities of the record, was made for each curve; this template was used in drawing a large

diagram and the figures here given are copies of these diagrams very much reduced. A few of the larger records were redrawn from measurements made on the photograph itself, but all the smaller ones, and most of the others, were pricked through under a lens by a very fine needle point and the record itself was then placed directly in front of the condensing lenses of a large projecting apparatus and thrown up on paper tacked to a screen; it was generally possible to see the whole of the diagram on the screen besides the bright images of the needle holes, and to reproduce this diagram much enlarged upon the paper. With a given battery, and a given current with a given

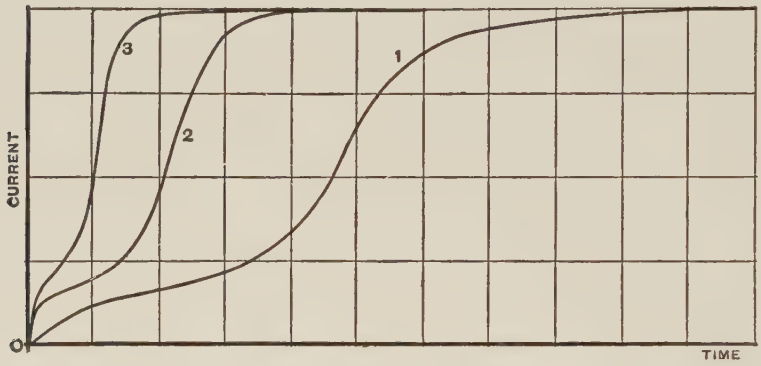


FIGURE 12

condition of the iron, it was always easy to get any desired number of records which, when superposed upon the screen, were practically indistinguishable.

Figure 11 shows a series [1] of building-up curves from a 15-kilowatt transformer very kindly placed at my disposal by Mr. S. E. Whiting. The finely laminated core of this transformer has a cross-section area of 108 square centimeters. The same magnetizing coil of about 340 turns was used throughout, but the electromotive force of the storage battery in the coil circuit could be changed so as to give the current the desired strength. In curve (1) the final intensity of the current was about 3 amperes, in curves (2) and (3) it was 1.5 amperes, and in curves (4) and (5), 0.75 amperes. With each value of the electromo-

[1] For a large number of similar curves, see Professor T. Gray's paper in *Phil. Trans.*, vol. clxxxiv.

tive force the steady current was sent through the coil, first in one direction and then in the other, for a number of times, and was then interrupted for a few moments previous to taking the photographic record from the oscillograph. Curves (1), (2), and (4) were obtained when the current in the coil had a direction opposite to that of the next preceding current; curves (3) and (5), when the current had the same direction as the preceding one. For a current of 0.37 amperes the re-

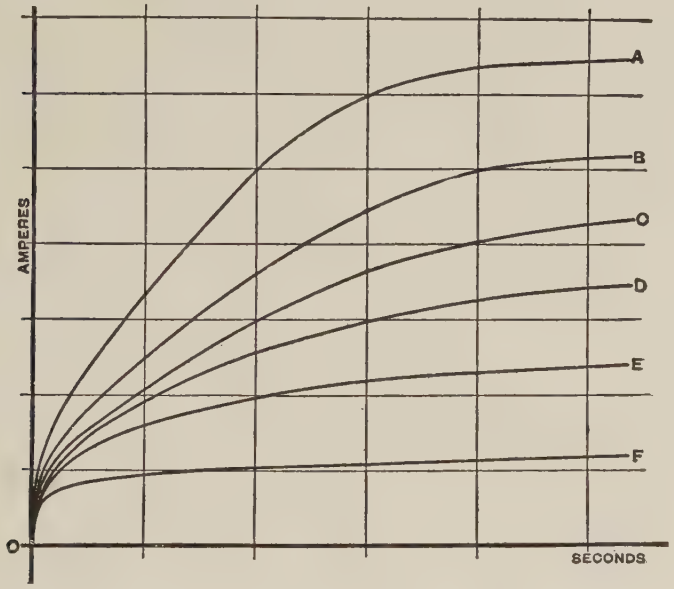


FIGURE 13

versed-current-curve had lost its points of inflection, and had become everywhere convex upward. With a voltage of only 6 and the low resistance primary of the transformer as exciting coil, the building-up time was an extremely short fraction of a second, and the building-up curve looked like a very straight, and nearly vertical, sign of integration.

If a number of equal coils of wire of a given size, each of resistance r, be wound together uniformly about a wooden ring so as to have equal self-inductances, and if a storage battery of small internal resistance be made to send a current through (say) n of the coils in series, the

inductance of the circuit will be nearly proportional to n^2 and the time-constant which is independent of the applied electromotive force and therefore of the current, will be nearly proportional to n. If the core of the ring be made of iron, the problem will, of course, be complicated in many ways, but in this connection Figure 12 is interesting; curves (3), (2), and (1) show the manner of growth of a current in coils of about 85, 170, and 340 turns about the core of the transformer just mentioned, in terms of its final value. The electromotive force was in reality the same for all three cases, and the currents were 6 amperes, 3 amperes, and 1.5 amperes.

The Manner of Growth of a Current in the Coil of an Electromagnet which has a Solid Core

We may now consider some oscillograph records which show the manner in which under given circumstances a current will grow in the

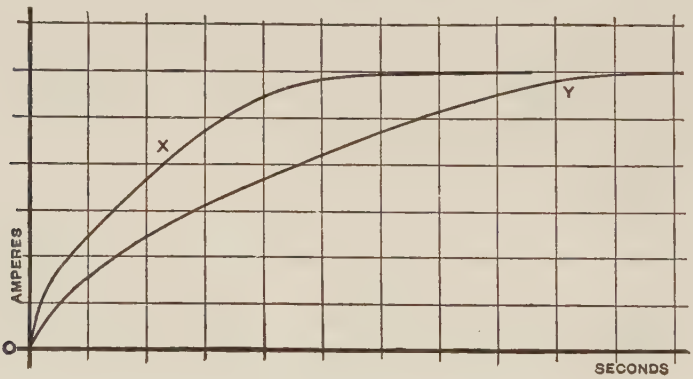


Figure 14

coil of the magnet represented by Figure 1. It is evident from what precedes that when other conditions are determined, the magnetic state of the core at the time when the coil circuit is closed will generally influence the result greatly. By sending through the coil a long series of steady currents of gradually decreasing intensity alternately in one direction and the other, one may reduce very low the residual magnetism in the iron even when the gap is closed.

Figure 13 shows building-up curves from a nearly neutral core,

under a voltage of about 84, when the air gap was closed, for currents of six different intensities from 1.2 amperes to 6.5 amperes.

Figure 14 shows curves obtained with a voltage of about 80 under two different conditions of the core. After the iron had been pretty well demagnetized, the growth of the current when the coil circuit was suddenly closed followed the law indicated by the upper curve: if, when the current had become steady, the circuit was broken and,

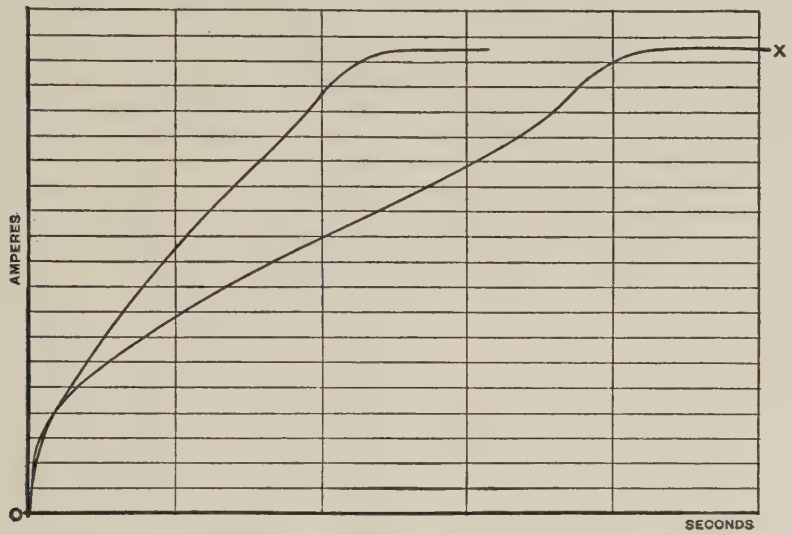


FIGURE 15

after fifteen or twenty seconds, closed with the direction of the current reversed, the march of the current followed the lower curve. The width of the gap was 1.6 mm. in this experiment. To study the influence of the width of the air gap upon the shape of the building-up curve of the current in the coil circuit I used first a storage battery of relatively high voltage (270) and obtained a long series of records for gap-widths up to about 30 mm. When the gap was closed and the current had been sent first in one direction and then in the other alternately for a number of times, I got the results given in Table XI for the growth of a current (A) which had the same direction as the next preceding current, and the results given in Table XII for a current (B) which had a direction opposite to that of the preceding current.

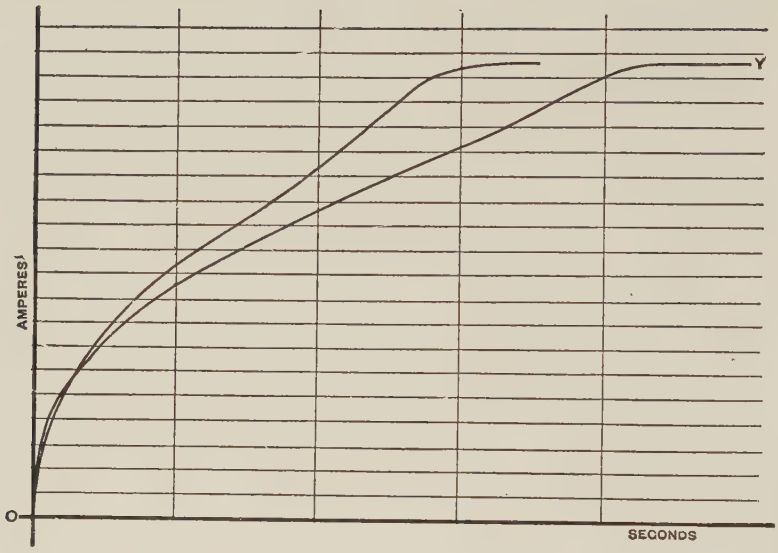


FIGURE 16

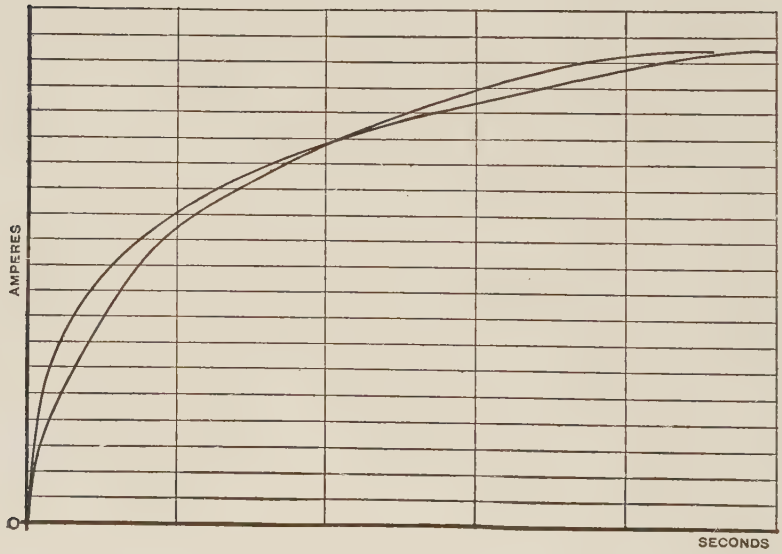


FIGURE 17

TABLE XI

A. (Direct Current)

Time in Seconds after the Closing of the Circuit	Approximate Intensity of the Current in Amperes	Time in Seconds after the Closing of the Circuit	Approximate Intensity of the Current in Amperes
0.10	3.20	1.50	13.95
0.20	4.36	1.75	15.41
0.30	5.37	1.80	15.84
0.40	6.33	1.90	16.42
0.50	7.21	2.10	17.67
0.75	9.07	2.20	18.16
1.00	10.96	2.40	18.56
1.25	12.55	2.50	18.60

When the gap is relatively narrow the shapes of the curves for direct and reversed currents differ little from those of the corresponding

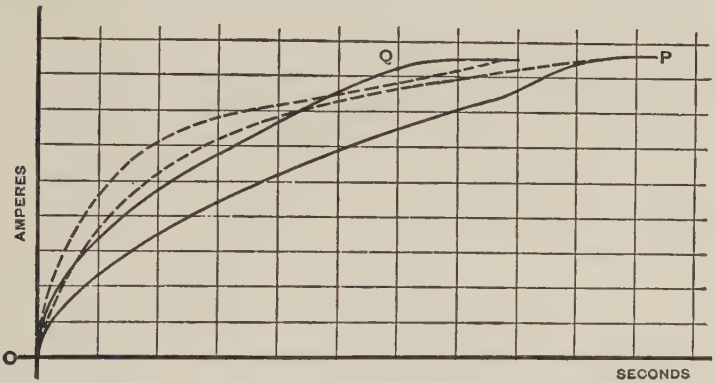


Figure 18

curves when the gap is closed, though the time required after the closing of the circuit for the current to attain (say) 99 per cent of its final value slowly increases with the gap-width. Figures 15, 16, 17 show curves for direct and reversed currents when the width of the air gap was 1.6 mm., 6.6 mm., and 28.3 mm. For gaps much wider than the last, the two curves of each diagram fall closely together. When with the same battery a sufficiently large non-inductive resistance was introduced into the circuit to reduce the current to 8.5 amperes, the building-up curves for gaps of width 1.6 mm. (full lines) and 25.0 mm. (broken lines) had the forms shown in Figure 18.

When the voltage of the storage battery was reduced to 90 the building-up curve for a reversed current of 6.9 amperes had the form of the

TABLE XII

B. (Reversed Current)

Time in Seconds after the Closing of the Circuit	Approximate Intensity of the Current in Amperes	Time in Seconds after the Closing of the Circuit	Approximate Intensity of the Current in Amperes
0.10	3.34	3.20	14.76
0.20	4.36	3.40	15.43
0.30	4.94	3.50	15.84
0.40	5.48	3.60	16.48
0.50	5.87	3.70	17.15
1.00	7.99	3.80	17.58
1.50	9.75	4.00	18.31
2.00	11.19	4.30	18.56
3.00	14.12	4.50	18.60

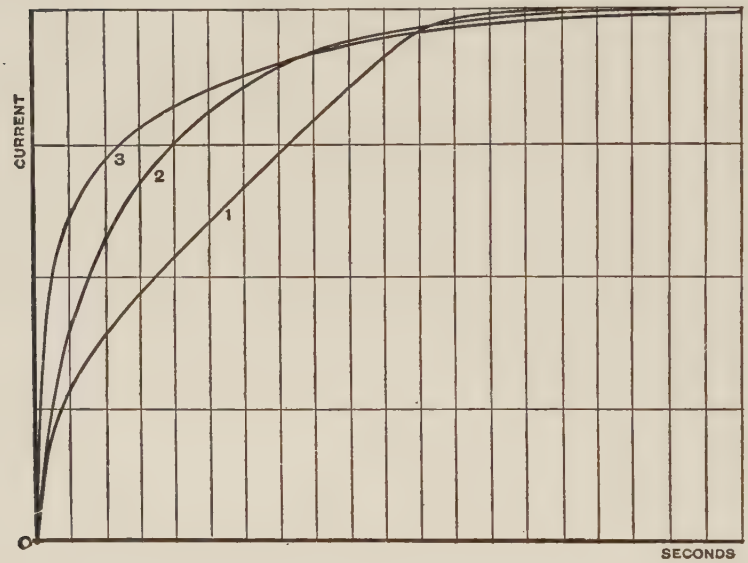


Figure 19

right-hand curve (1) in Figure 19 when the gap was closed, but of the next curve (2) when the air gap was 13 mm. wide. The curves cross eleven seconds after the start. For the current to attain half its final strength, 3.5 seconds are required when the gap is closed, but only 1.5 seconds when the width of the gap is 13 mm.; the current attains 99 per cent of its final value more quickly when the gap is closed than when it is open.

The leftmost curve (3) of Figure 19 belongs to a reversed current of

1.3 amperes when the gap was closed; the ordinates are exaggerated so as to make the final value the same as for the larger current.

When a secondary coil, consisting of a few turns of insulated wire wound around the whole core of the magnet, was connected with an oscillograph which made its record on the same sheet of paper as the oscillograph connected with the main circuit, it was easy to get the rate of growth of the induction flux in the core. Figure 20 shows building-up curves when the gap was closed for a current of 1.3 amperes furnished (*A*) by a battery of 40 storage cells and (*D*) by a battery of 10 storage cells; in the first case it was necessary of course to add non-inductive resistance to the circuit to reduce the current to

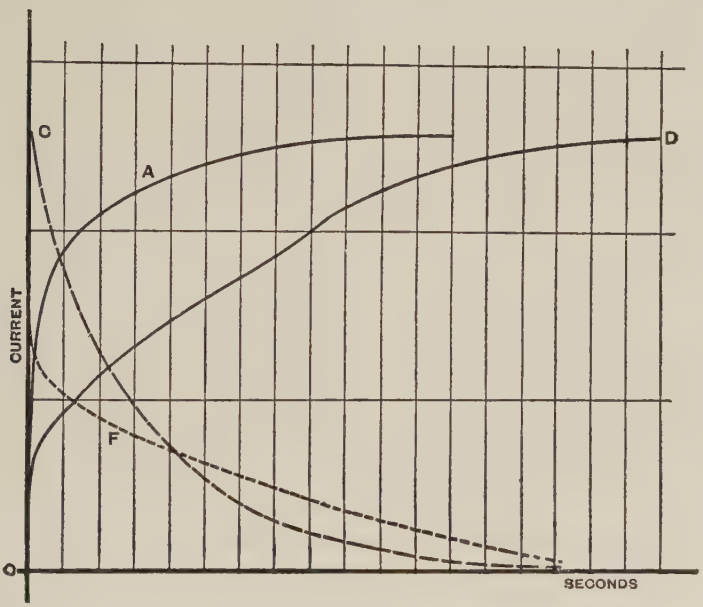


FIGURE 20

this value. The time required for the current to attain practically its final value is far less in the first case than in the second. The dotted curves show the records of the oscillograph in the secondary circuit on an arbitrary scale. The two curves are quite unlike in shape, but the areas under them, as measured by a planimeter, are almost exactly the same. The general forms of the dotted curves here shown are like all the scores of others which I have obtained under all sorts of conditions of current strength and resistance.

The pole pieces of the magnet shown in Figure 1 are fastened to the cylindrical arms by long bolts which extend outside the frame and carry nuts which press upon the yokes and serve to keep the jaws apart. When one of these bolts was removed it left a long axial hole of about an inch in diameter through the arm, and into this I put a long rod of soft iron upon which a layer of fine insulated wire had been wound pretty uniformly, and this coil was connected with the secondary oscillograph already mentioned. This coil and its long core filled the cavity fairly completely except at the outside ends, but doubtless the joint at the inner end of the core affected the results somewhat, and in a manner not easy to treat mathematically; nevertheless the indications of this coil were interesting and characteristic in their general features, as we shall see hereafter, of the behavior of the eddy currents in the core near its axis. When a current was started in the coil circuit, the mirror of the oscillograph attached to the secondary coil remained sensibly at rest for an interval the length of which depended upon the final value of the coil current, the electromotive force in its circuit, and the magnetic condition of the core at the start; then a temporary current which had a sensible value for perhaps ten or fifteen seconds passed through the secondary, though if a sensitive ballistic galvanometer instead of the oscillograph had been connected with the coil it would have been made clear that the temporary current had not wholly disappeared at the end of this interval. When, after the current in the main circuit had been apparently steady for a minute or two the coil circuit was suddenly broken, the current in the secondary became almost immediately evident, soon attained its maximum value, and then died slowly away so that the mirror seemed to reach its zero again after about thirty seconds.

Figure 21 shows a typical oscillograph record accurately reproduced. It was obtained with the air gap closed and with a current of 3.12 amperes in the main circuit from a storage battery of about 84 volts. For five or six seconds after the start there was no sensible indication in the secondary coil, though by that time the main current (1) had attained about three fourths of its final value. The greatest value of the current in the secondary did not occur until rather more than fourteen seconds after the current in the primary had begun, and the secondary current had apparently died out in less than thirty sec-

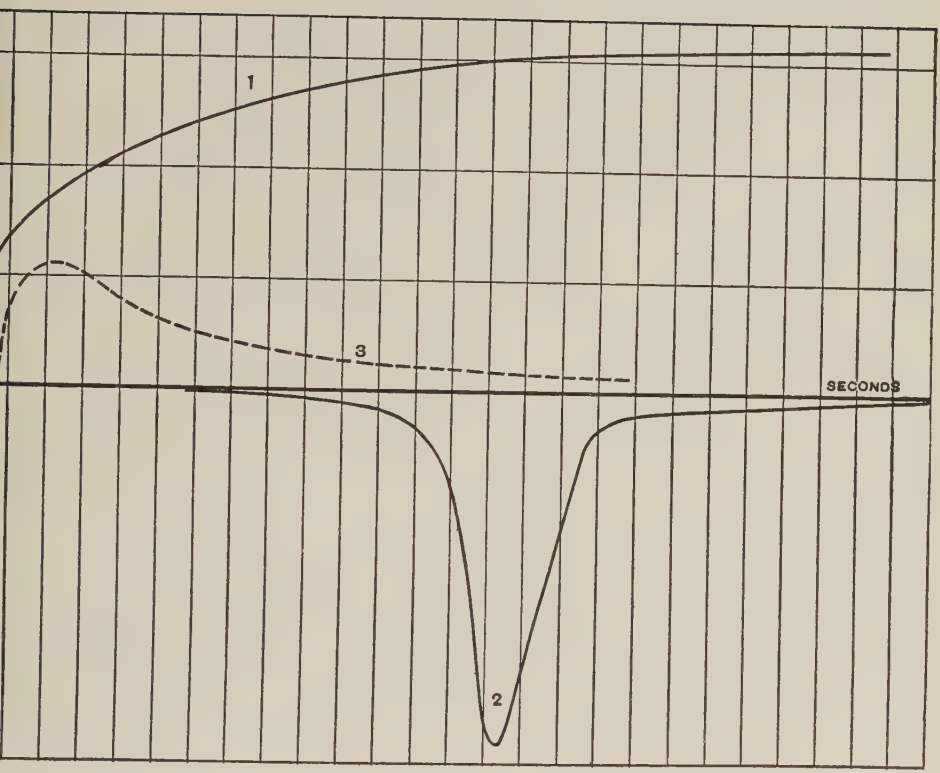


FIGURE 21

onds from the beginning. The ordinates of the curve of the current in the main circuit represent amperes; the ordinates of the curve (2) which represents the current in the secondary, are on an arbitrary scale. If at the time $t = 0$, the main circuit through which a steady current of 3.12 amperes had been passing were suddenly broken, the current in the secondary would have grown and died out after the manner indicated by the dotted curve (3). For about half a second after the main circuit was interrupted, there would have been no sensible current in the secondary, but then it would have suddenly appeared in a manner that strongly suggests the theoretical curves shown in Figures 6 and 8. Figure 22 illustrates the fact that the form of the secondary current curve depends very much upon the intensity of the current in the main circuit, and that sometimes a comparatively slight change in the latter will alter materially the maximum intensity of the secondary current. The diagram is a careful reproduction of the records of the secondary oscillograph for currents of 3.76 amperes, 3.12 amperes, and 2.60 amperes respectively. The scale at the bottom represents seconds; the records, which are to be read from right to left, are displaced with respect to each other merely to prevent confusion in the figure.

The lag in seconds of the crest of the current in the secondary circuit behind the closing of the main circuit depends, naturally enough, upon the final intensity of the main current and upon the magnetic condition of the core at the outset. If a series of currents of the same final intensity be passed first in one direction and then in the other through the main coil, and if after the circuit has been broken for half a minute the current be sent through the coil again, the retardation of the crest of the secondary will be perhaps twice or thrice as great if the direction of the new current be opposed to that of the next preceding one, as it would be if the new current had the same direction as its predecessor.

In Figure 23 the abscissas represent currents in amperes and the ordinates the lag, in tens of seconds, of the crest of the secondary current behind the closing of the primary circuit when the voltage was about 84 and when the current considered had the opposite direction to the one before it. The gap was in this case closed. For a current of 1 ampere, as the diagram shows, the lag is about 40 seconds, while

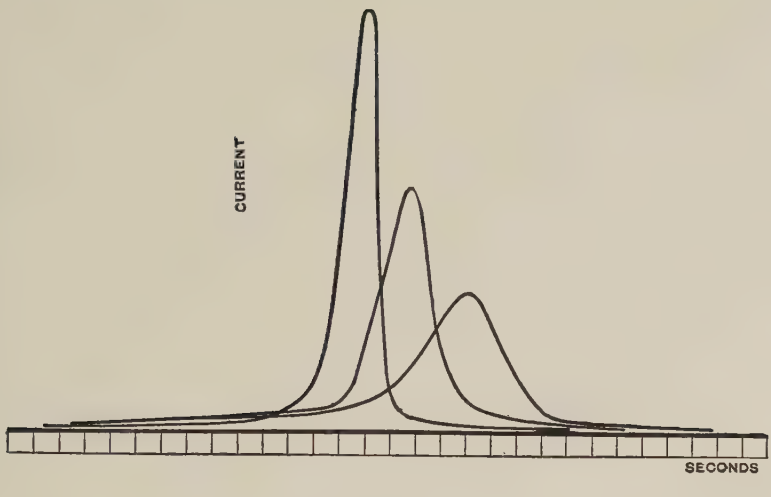


FIGURE 22

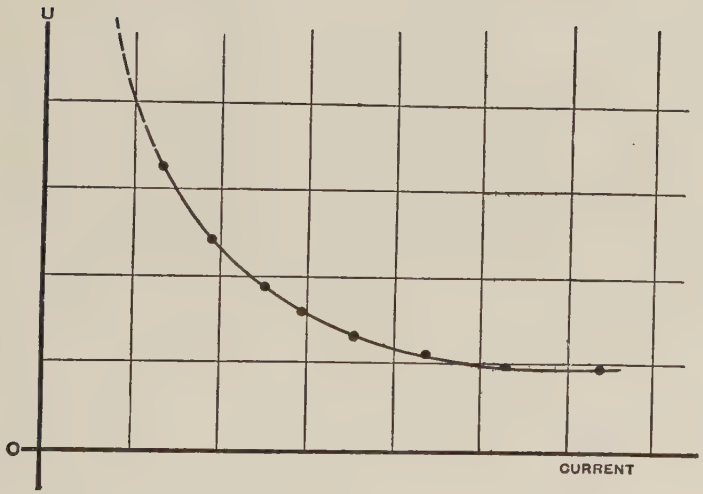


FIGURE 23

for a current of 5 amperes it is only 10 seconds. In Figure 24 the ordinates represent the relative heights of the crests of the secondary cur-

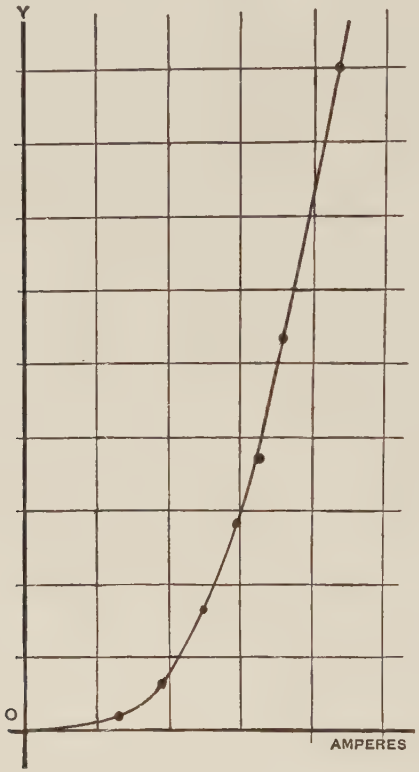


FIGURE 24

rents corresponding to primary currents the intensities of which, in amperes, are represented by the abscissas.

In the next portion of this paper I hope to discuss some of the results given here, with a large number of others which throw some light upon the march of the eddy currents in a massive iron core of the kind here used. In this connection the records of the secondary coils wound upon the separate members of a set of loosely-fitting, coaxial, cylindrical shells of soft steel, made part of the core of the magnet represented in Figure 1, will be interesting.

VI

ON THE PERMEABILITY AND THE RETENTIVENESS OF A MASS OF FINE IRON PARTICLES [1]

In a familiar lecture-room experiment, a mass of iron filings, filling a straight glass tube to a length of thirty or forty times its diameter, is forced to become a rather weak "bar magnet" by subjecting it to a strong exciting field in a solenoid; and then, by rearranging the particles, it is made to lose its magnetic moment almost completely, although, if after the filings have been poured out of the tube, a few of them be examined under a microscope of moderate power, it is usually easy to see that most of the elongated particles retain some magnetism for a good while. A number of persons [2] have studied the magnetic properties of masses of iron filings or of chemically deposited "iron dust," as well as of mixtures of iron particles in various proportions with nonmagnetic powders of various kinds. Concise statements of the results of experiments on the subject are to be found in the papers of Maurain and Trenkle.

I have lately had occasion to measure the permeability and the retentiveness of each of several masses of very fine cast-iron particles, or dust, made by a fine cutting end-mill in a milling-machine, and, as testing the effect upon their magnetic properties of subjecting the particles to a "hardening" process, the results seem to have some slight interest.

The material to be examined was tamped solidly into a glass tube almost exactly one centimeter in diameter, until a column was formed fifty diameters long. After its ends had been closed by corks, the comparatively short tube was put into the middle of a long solenoid S, the axis of which was horizontal and perpendicular to the meridian. The

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlii, no. 3, June, 1906.

[2] Töpler, *Pogg. Ann. d. Phys.*, 1877; v. Waltenhofen, *Wied. Ann. d. Phys.*, 1870; Jamin, *Comptes rendus*, 1875; Börnstein, *Pogg. Ann. d. Phys.*, 1875; Auerbach, *Wied. Ann. d. Phys.*, 1880; Baur, *Wied. Ann. d. Phys.*, 1880; Maurain, *Éclairage, Élect.*, 1903; Trenkle, *Erlangen, Sitzungsberichte*, 1905; *Wied. Ann. d. Phys.*, 1906.

connections of the apparatus are shown schematically in Figure 1. After the tube (Q) had been placed in S, the double switch T was closed to the left, so as to connect S with the secondary (L) of a transformer the primary of which was attached to the alternating street circuit. The secondary coil was so suspended with counterbalancing weights in a tall frame that it could be moved at will in the direction of its axis to a distance of several feet away from the primary, and thus a great number of alternations of a current gradually decreasing in intensity could be sent through S to demagnetize the specimen in it.

M represents a magnetometer in the form of a mirror galvanometer placed in Gauss's B Position with respect to Q; the galvanometer was so shunted by an adjustable resistance X that the effect on the galvanometer needle of the partial current through the coils of the instrument almost exactly compensated for the effect of the whole current through the solenoid S when empty. R is an adjustable rheostat of 200 ohms total resistance, designed to carry currents of some intensity, K is a commutator, and W a milliamperemeter furnished with a set of four shunts. When the switch T was closed to the right, it was possible, by manipulating the rheostat arm and the commutator K, to put Q through any desired hysteresis cycle in the usual manner. The current came from a battery of storage cells, any number of which could be used at pleasure. A current of 1 ampere in the solenoid gave rise to a field of 54.8 gausses in the space within it. The field about M's needle had to be artificially strengthened to suit the circumstances, and a piece of soft Bessemer steel rod of almost exactly the same dimensions as the column of filings was used to determine the sensitiveness of the apparatus at any time. By means of a coil of 20 turns of extremely fine insulated copper wire wound directly on this piece at its centre and connected with a carefully standardized ballistic galvanometer, the induction flux through the centre of the rod could be found and the corresponding deflection of the magnetometer needle determined.

The work was undertaken in order to compare the magnetic properties of masses of iron particles, as they came from the milling-machine, with those of masses of particles from the same lot which had undergone the treatment used in hardening iron castings for permanent magnets. These "filings" were prepared by Mr. G. W.

Thompson, the mechanician of the Jefferson Laboratory, who has had much experience in the process. A completely closed iron crucible with thin walls, containing a mass of the particles to be treated, was

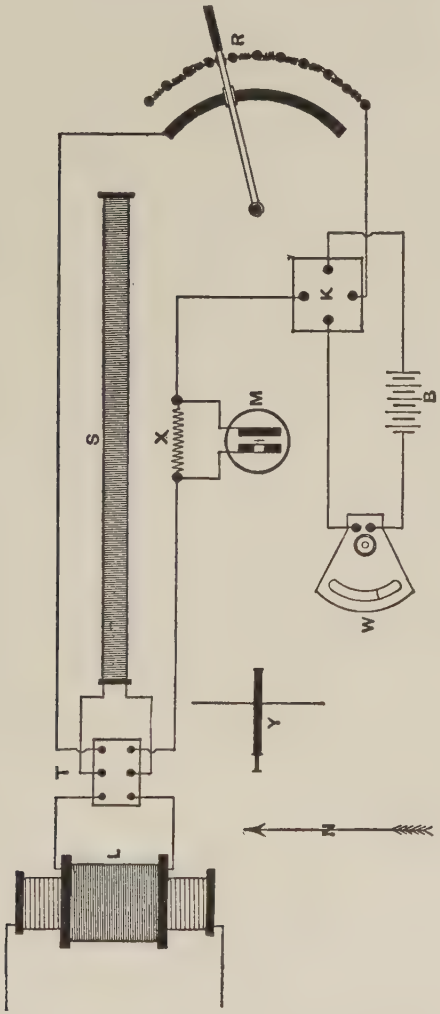


FIGURE 1

heated white hot under a power blast in a gas furnace, and then suddenly chilled in an acid bath. After this experience, during which the particles had been carefully excluded from the air, they had a somewhat altered color and lustre, but under a microscope of low power

showed very little difference from the untreated particles; at best all such particles cut by machine tools from iron castings are most irregular in form, and are so much seamed by deep furrows and pits as to look like clinkers from a furnace. All the particles were kept quite free from oil or dirt, and the surfaces of the "hardened" ones were only very slightly tarnished, but it was not possible to pack quite so large a mass of the material into a given space after the treatment as

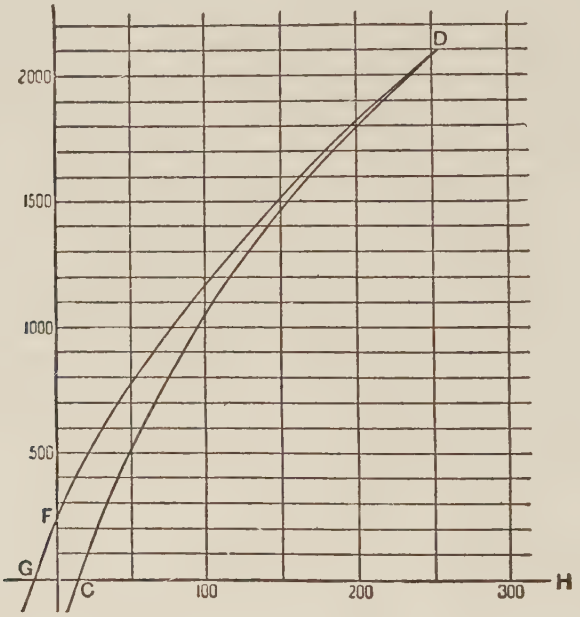


FIGURE 2

before. This might have arisen from changes of shape, but it is a suspicious fact that the induction flux through the center of a column (of given dimensions) of the filings under a given excitation was almost exactly the same whether the filings had been hardened or not. Two uniform glass tubes from the same piece and practically of the same dimensions were filled respectively with 103 grams of untreated particles and 96 grams of the others, and each was put several times through a hysteresis cycle, using about 250 gausses as the intensity of the maximum exciting field. After this 22 simultaneous determinations of H and B were made in each half-cycle, and both cycles were

carefully plotted on such a scale that each was about 40 centimeters long. It appeared that the maximum values of the induction were almost identical, and that at no point were the plotted curves so much as 1 millimeter apart. In filling the same glass tube a number of times in succession from the same lot of filings, it was of course impossible to pack exactly the same mass into the same space twice; but the hysteresis diagrams, many of which were obtained by Mr. J. Coulson and myself, were always of the same shape, and the intensity of magnetization due to a given exciting field seemed to be strictly proportional to the mass. The density of the untreated filings was only about four tenths of that of massive cast iron.

The demagnetizing effect of the ends in a rod of solid iron only fifty diameters long would of course be very serious, but in the present case, due to the relatively small value of I for a given value of H, it is far less important. In order to determine from my observations the permeability at the center of an endless column of material like mine, I have used the value of N computed by Du Bois [1] from the experimental results of Ewing, Tanakadaté, and himself.

Figure 2 shows half of a hysteresis diagram obtained by shearing very slightly a diagram obtained by Mr. J. Coulson and me from a column of untreated particles 50 centimeters long and 1 centimeter in diameter, which weighed just over 100 grams. This diagram has all the characteristics of the many others for which I have the materials.

As the figure shows, B was almost exactly 2100 for an exciting field of 255 gausses, and this corresponds to a value for I of only about 147. When the external field was removed, the remanent value of I was about 20.8; the coercive force, however, was comparatively large, being about 16. It is evident that the exciting field would need to be very strong to magnetize the column of particles approximately to saturation. The subject of the saturization of masses of iron filings has been discussed at length by Maurain and by Trenkle.

[1] Ewing, *Philosophical Transactions;* Tanakadaté, *Philosophical Magazine,* 1888; Du Bois, *Annalen der Physik.*, 1892; *The Magnetic Circuit*, chapter vi. For a criticism on this process, see Trenkle, *loc. cit.*

VII

ON THE CONDITIONS TO BE SATISFIED IF THE SUMS OF THE CORRESPONDING MEMBERS OF TWO PAIRS OF ORTHOGONAL FUNCTIONS OF TWO VARIABLES ARE TO BE THEMSELVES ORTHOGONAL [1]

If $\phi_1(x, y)$, $\phi_2(x, y)$ are the potential functions due to two columnar distributions of matter the lines of which are perpendicular to the xy plane, and if $\psi_1(x, y)$, $\psi_2(x, y)$ are conjugate to ϕ_1 and ϕ_2, respectively, the families of curves obtained by equating ψ_1 and ψ_2 to parameters, are lines of force of the two distributions. Moreover, $\phi_1 + \phi$ is the potential function due to a combination of the two distributions, and the function $\psi_1 + \psi_2$ equated to a parameter gives the corresponding lines of force. The fact that if (ϕ_1, ψ_1) are any pair of conjugate functions and (ϕ_2, ψ_2) any other such pair, the functions $(a\phi_1 + b\phi_2, a\psi_1 + b\psi_2)$ are also conjugate — with similar facts for other classes of functions — lies at the foundation of the graphical methods so successfully used by Maxwell [2] and by others in drawing equipotential lines, and lines of force or flow, due to combinations of simple elements. If (ϕ_1, ψ_1) are merely a pair of orthogonal functions and (ϕ_2, ψ_2) another such pair, it is generally not true that $(\phi_1 + \phi_2, \psi_1 + \psi_2)$ are an orthogonal pair: thus (x, y), $(x^2 + y^2, y/x)$ are pairs of orthogonal functions, but $x + x^2 + y^2$, $y + y/x$ are not orthogonal.

In certain classes of physical problems one encounters potential functions which are not themselves harmonic and the lines of which are not possible lines of any harmonic function, and it is often desirable in cases where the analytical processes become too complex, to determine graphically the forms of lines of force or flow due to a combination of two simple elements. This note discusses briefly the conditions under which the ordinary method of procedure is possible.

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlii, no. 7, July, 1906.

[2] Maxwell, *Treatise on Electricity and Magnetism*, vol. i, ch. 7; Minchin, *Uniplanar Kinematics*, § 112. See also P. W. Bridgman, "The Electrostatic Field surrounding two special Columnar Elements," *Proceedings of the American Academy*, xli, 28.

Let (α, β) and (λ, μ) be two pairs of orthogonal functions of the two variables (x, y) so that

$$\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \beta}{\partial y} = 0, \tag{1}$$

$$\frac{\partial \lambda}{\partial x} \cdot \frac{\partial \mu}{\partial x} + \frac{\partial \lambda}{\partial y} \cdot \frac{\partial \mu}{\partial y} = 0; \tag{2}$$

then if $(\alpha + \lambda, \beta + \mu)$ are to form an orthogonal pair, the equation

$$\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \lambda}{\partial x}\right)\left(\frac{\partial \beta}{\partial x} + \frac{\partial \mu}{\partial x}\right) + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \lambda}{\partial y}\right)\left(\frac{\partial \beta}{\partial y} + \frac{\partial \mu}{\partial y}\right) = 0 \tag{3}$$

must be identically satisfied. Since (1) and (2) are true, (3) takes the form

$$\left(\frac{\partial \lambda}{\partial x} \cdot \frac{\partial \beta}{\partial x} + \frac{\partial \lambda}{\partial y} \cdot \frac{\partial \beta}{\partial y}\right) + \left(\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \mu}{\partial x} + \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \mu}{\partial y}\right) = 0. \tag{4}$$

If h_α, h_β, h_λ, h_μ represent the values of the gradients of α, β, λ, μ, and if the angle at any point between the directions in which λ and β increase most rapidly be denoted by $[\lambda, \beta]$, (4) becomes

$$h_\lambda \cdot h_\beta \cdot \cos [\lambda, \beta] + h_\alpha \cdot h_\mu \cdot \cos [\alpha, \mu] = 0. \tag{5}$$

Whatever the sequence of the directions of the gradient vectors might be, the two angles which appear in (5) would be either equal or supplementary, and their cosines would be equal in absolute value, but the gradients themselves are intrinsically positive and the sequences must therefore be such that

$$h_\lambda / h_\mu = h_\alpha / h_\beta. \tag{6}$$

Suppose that in the case of two given pairs of orthogonal functions (α, β) (λ, μ), the necessary condition (6) is satisfied, and that the value of the gradient ratio, h_α / h_β, obtained from the given values of α and β, is the function Ω of x and y; then

$$\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 = \Omega^2 \left(\frac{\partial \beta}{\partial x}\right)^2 + \Omega^2 \left(\frac{\partial \beta}{\partial y}\right)^2, \tag{7}$$

and if, for $(\partial \alpha)^2/(\partial y)^2$ in this equation, we substitute the value

$$\left(\frac{\partial \alpha}{\partial x}\right)^2 \left(\frac{\partial \beta}{\partial x}\right)^2 \Big/ \left(\frac{\partial \beta}{\partial y}\right)^2 \tag{8}$$

obtained from (1), it appears, since the gradient of a real function cannot vanish, that

$$\frac{\partial \alpha}{\partial x} = \pm \Omega \left(\frac{\partial \beta}{\partial y} \right). \tag{9}$$

If for $(\partial \alpha)^2/(\partial x)^2$ in (7), we substitute its equivalent

$$\left(\frac{\partial \alpha}{\partial y} \right)^2 \left(\frac{\partial \beta}{\partial y} \right)^2 \Big/ \left(\frac{\partial \beta}{\partial x} \right)^2 \tag{10}$$

derived from (1), we shall learn that

$$\frac{\partial \alpha}{\partial y} = \mp \Omega \left(\frac{\partial \beta}{\partial x} \right). \tag{11}$$

Either the upper signs or the lower signs must be used in (9) and (11).

If now we treat the equation

$$h_\lambda{}^2 = \Omega^2 h_\mu{}^2 \tag{12}$$

in a similar way we shall obtain the equations

$$\frac{\partial \lambda}{\partial x} = \pm \Omega \left(\frac{\partial \mu}{\partial y} \right), \tag{13}$$

$$\frac{\partial \lambda}{\partial y} = \mp \Omega \left(\frac{\partial \mu}{\partial x} \right), \tag{14}$$

and, so far as the relation (12) is concerned, we may use either the upper signs or the lower signs, but if (4) is to be identically satisfied, the *same* sign must be used in (9) and (13) and the sign opposite to this in (11) and (14). Equation (6), then, together with the proper choice of sequence of directions for the gradient vectors which corresponds to the convention with regard to signs just made, will ensure the orthogonality of $\alpha + \lambda$, $\beta + \mu$. For practical purposes, however, it is well to approach the problem from another side.

If (α, β) and (λ, μ) are given pairs of orthogonal functions, and if we denote the given scalar point functions obtained by dividing $\partial\beta/\partial x$ by $\partial\alpha/\partial y$, and by dividing $\partial\mu/\partial x$ by $\partial\lambda/\partial y$, by ζ and η, the equations (1) and (2) can be written in the forms

$$\frac{\partial \beta}{\partial x} \Big/ \frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial y} \Big/ \frac{\partial \alpha}{\partial x} = \zeta, \tag{15}$$

and
$$\frac{\partial \mu}{\partial x} \Big/ \frac{\partial \lambda}{\partial y} = -\frac{\partial \mu}{\partial y} \Big/ \frac{\partial \lambda}{\partial x} = \eta, \tag{16}$$

or
$$\frac{\partial \beta}{\partial x} = \zeta \cdot \frac{\partial \alpha}{\partial y}, \qquad \frac{\partial \beta}{\partial y} = -\zeta \cdot \frac{\partial \alpha}{\partial x}, \tag{17}$$

and
$$\frac{\partial \mu}{\partial x} = \eta \cdot \frac{\partial \lambda}{\partial y}, \qquad \frac{\partial \mu}{\partial y} = -\eta \cdot \frac{\partial \lambda}{\partial x}. \tag{18}$$

If the values of the derivatives of β and μ given in (17) and (18) be substituted in (4) this equation becomes

$$(\zeta - \eta)\left(\frac{\partial \lambda}{\partial x} \cdot \frac{\partial \alpha}{\partial y} - \frac{\partial \alpha}{\partial x} \cdot \frac{\partial \lambda}{\partial y}\right) = 0, \tag{19}$$

and if $(\alpha + \lambda, \beta + \mu)$ are to be orthogonal, α and λ must be such as to satisfy it. If λ were expressible as a function of α, and μ as a function of β, the second factor would vanish, but this case is of no practical interest and (19) demands in general that ζ and η shall be identical, so that

$$\frac{\partial \beta}{\partial x} = \zeta \cdot \frac{\partial \alpha}{\partial y}, \qquad \frac{\partial \beta}{\partial y} = -\zeta \cdot \frac{\partial \alpha}{\partial x}, \tag{17}$$

and
$$\frac{\partial \mu}{\partial x} = \zeta \cdot \frac{\partial \lambda}{\partial y}, \qquad \frac{\partial \mu}{\partial y} = -\zeta \cdot \frac{\partial \lambda}{\partial x}. \tag{20}$$

If in these equations the arbitrary function ζ is made equal to unity, the conditions degenerate into the familiar definitions of any two pairs of conjugate functions.

In order that a single function (β) may exist the partial derivatives of which with respect to x and y shall be equal, respectively, to

$$\zeta \cdot \frac{\partial \alpha}{\partial y} \text{ and } -\zeta \cdot \frac{\partial \alpha}{\partial x},$$

it is necessary and it is sufficient that α and ζ should satisfy the condition

$$\frac{\partial}{\partial y}\left(\zeta \cdot \frac{\partial \alpha}{\partial y}\right) + \frac{\partial}{\partial x}\left(\zeta \cdot \frac{\partial \alpha}{\partial x}\right) = 0, \tag{21}$$

or
$$\frac{\partial \zeta}{\partial x} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial \zeta}{\partial y} \cdot \frac{\partial \alpha}{\partial y} + \zeta \cdot \nabla^2(\alpha) = 0. \tag{22}$$

In order that μ may exist, ζ and λ must satisfy the equation

$$\frac{\partial \zeta}{\partial x} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial \zeta}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \zeta \cdot \nabla^2(\lambda) = 0. \tag{23}$$

If $\log_e \zeta$ be represented by $\bar{\omega}$, the last two equations take the forms

$$\frac{\partial\bar{\omega}}{\partial x}\cdot\frac{\partial\alpha}{\partial x}+\frac{\partial\bar{\omega}}{\partial y}\cdot\frac{\partial\alpha}{\partial y}+\nabla^2(\alpha)=0, \tag{24}$$

$$\frac{\partial\bar{\omega}}{\partial x}\cdot\frac{\partial\lambda}{\partial x}+\frac{\partial\bar{\omega}}{\partial y}\cdot\frac{\partial\lambda}{\partial y}+\nabla^2(\lambda)=0, \tag{25}$$

and, if each of these be differentiated with respect to x and with respect to y, $\bar{\omega}$ may be eliminated from the resulting equations and a necessary condition for α and λ obtained, which may be stated in the form of the determinantal equation —

$$\begin{vmatrix}
\frac{\partial\alpha}{\partial x} & \frac{\partial\alpha}{\partial y} & 0 & 0 & 0 & \nabla^2(\alpha)\\
\frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & 0 & 0 & 0 & \nabla^2(\lambda)\\
\frac{\partial^2\alpha}{\partial x^2} & \frac{\partial^2\alpha}{\partial x\cdot\partial y} & \frac{\partial\alpha}{\partial x} & \frac{\partial\alpha}{\partial y} & 0 & \frac{\partial}{\partial x}\left(\nabla^2(\alpha)\right)\\
\frac{\partial^2\lambda}{\partial x^2} & \frac{\partial^2\lambda}{\partial x\cdot\partial y} & \frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & 0 & \frac{\partial}{\partial x}\left(\nabla^2(\lambda)\right)\\
\frac{\partial^2\alpha}{\partial x\cdot\partial y} & \frac{\partial^2\alpha}{\partial y^2} & 0 & \frac{\partial\alpha}{\partial x} & \frac{\partial\alpha}{\partial y} & \frac{\partial}{\partial y}\cdot\left(\nabla^2(\alpha)\right)\\
\frac{\partial^2\lambda}{\partial x\cdot\partial y} & \frac{\partial^2\lambda}{\partial y^2} & 0 & \frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & \frac{\partial}{\partial y}\cdot\left(\nabla^2(\lambda)\right)
\end{vmatrix} = 0. \tag{26}$$

If α and λ happen to be harmonic, the elements of the last column vanish and the equation is satisfied, as it should be.

It is possible to factor the determinant, after it has been reduced, and if

$$\begin{aligned}
L &\equiv \frac{\partial^2\alpha}{\partial x^2}\cdot\frac{\partial\lambda}{\partial x}-\frac{\partial^2\lambda}{\partial x^2}\cdot\frac{\partial\alpha}{\partial x}+\frac{\partial^2\alpha}{\partial x\cdot\partial y}\cdot\frac{\partial\lambda}{\partial y}-\frac{\partial^2\lambda}{\partial x\cdot\partial y}\cdot\frac{\partial\alpha}{\partial y},\\
M &\equiv \frac{\partial^2\alpha}{\partial x\cdot\partial y}\cdot\frac{\partial\lambda}{\partial x}-\frac{\partial^2\lambda}{\partial x\cdot\partial y}\cdot\frac{\partial\alpha}{\partial x}+\frac{\partial^2\alpha}{\partial y^2}\cdot\frac{\partial\lambda}{\partial y}-\frac{\partial^2\lambda}{\partial y^2}\cdot\frac{\partial\alpha}{\partial y},\\
N &\equiv \frac{\partial}{\partial x}\left(\Delta^2(\alpha)\right)\cdot\frac{\partial\lambda}{\partial x}-\frac{\partial}{\partial x}\left(\Delta^2(\lambda)\right)\cdot\frac{\partial\alpha}{\partial x}+\frac{\partial}{\partial y}\left(\Delta^2(\alpha)\right)\cdot\frac{\partial\lambda}{\partial y}\\
&\qquad -\frac{\partial}{\partial y}\left(\Delta^2(\lambda)\right)\cdot\frac{\partial\alpha}{\partial y},
\end{aligned} \tag{27}$$

the condition of (26) demands either that α and λ satisfy the equation

$$\begin{vmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \nabla^2(\alpha) \\ \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} & \nabla^2(\lambda) \\ L & M & N \end{vmatrix} = 0, \tag{28}$$

or else that α and λ have the same level curves: this last case, as being uninteresting, may be left out of account.

Sometimes (28) is more convenient than the unexpanded form of the same condition which follows immediately if we solve (24) and (25) for $\partial\bar{\omega}/\partial x$ and $\partial\bar{\omega}/\partial y$,

$$\frac{\partial \bar{\omega}}{\partial x} = \frac{\frac{\partial \alpha}{\partial y} \cdot \nabla^2(\lambda) - \frac{\partial \lambda}{\partial y} \cdot \nabla^2(\alpha)}{\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \lambda}{\partial y} - \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \lambda}{\partial x}}, \tag{29}$$

$$\frac{\partial \bar{\omega}}{\partial y} = \frac{\frac{\partial \lambda}{\partial x} \cdot \nabla^2(\alpha) - \frac{\partial \alpha}{\partial x} \cdot \nabla^2(\lambda)}{\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \lambda}{\partial y} - \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \lambda}{\partial x}}, \tag{30}$$

and equate the derivative with respect to y of the second member of the first equation to the derivative with respect to x of the second member of the other. This process yields the relation,

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial \alpha}{\partial y} \cdot \nabla^2(\lambda) - \frac{\partial \lambda}{\partial y} \cdot \nabla^2(\alpha)}{\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \lambda}{\partial y} - \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \lambda}{\partial x}}\right] = \frac{\partial}{\partial x}\left[\frac{\frac{\partial \lambda}{\partial x} \cdot \nabla^2(\alpha) - \frac{\partial \alpha}{\partial x} \cdot \nabla^2(\lambda)}{\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \lambda}{\partial y} - \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \lambda}{\partial x}}\right] \tag{31}$$

and it is possible to check the fact that (28) and (31) are equivalent by a straightforward but somewhat laborious comparison of the two.

If α and λ satisfy (31), a function ζ exists which satisfies (22) and (23), functions β and μ exist which satisfy (17) and (20), and (α, β), (λ, μ), $(\alpha + \lambda, \beta + \mu)$ are orthogonal pairs of functions.

If, for instance, both α and λ represent values in the xy plane of harmonic space functions (V, W) the level surfaces of which are sur-

faces of revolution about the x axis, so that

$$\frac{1}{y}\cdot\frac{\partial}{\partial y}\left(y\cdot\frac{\partial V}{\partial y}\right)+\frac{\partial^2 V}{\partial x^2}=0 \tag{32}$$

with a similar equation for W,

$$\nabla^2(\alpha)=-\frac{1}{y}\cdot\frac{\partial \alpha}{\partial y},\quad \nabla^2(\lambda)=-\frac{1}{y}\cdot\frac{\partial \lambda}{\partial y}, \tag{33}$$

equation (31) is satisfied and

$$\frac{\partial \bar{\omega}}{\partial x}=0,\quad \frac{\partial \bar{\omega}}{\partial y}=\frac{1}{y},\quad \zeta=cy. \tag{34}$$

In this case, if we put $c=1$, β and μ are the Stokes functions corresponding to α and λ. If the level surfaces of the harmonic space functions, V and W, are surfaces of revolution about two different straight lines in the xy plane, the functions α and λ which represent the values of V and W in this plane do not in general satisfy (31).

Graphical superposition of the lines of force in the xy plane due to an infinitely long, homogeneous cylinder of revolution parallel to the axis, and to a homogeneous sphere with centre in the plane, will not in general yield the lines of force in the xy plane due to a combination of the two masses.

If α and λ are harmonic, any linear function (but no other than a linear function) of α is harmonic, and any two linear functions of α and λ satisfy (31). There generally exist, however, non-linear functions of α and λ which, although they are not harmonic, satisfy the condition. The functions (x^2-y^2), $(x^2+y^2)^n$, the second of which is not harmonic, obey (31), as do the harmonic pair (x^2-y^2), $\log(x^2+y^2)$.

As a simple example of the fact that a harmonic function and a function which is not even isothermal may satisfy the condition (31), we may consider $(2y^2-x^2)$ and (y^2-x^2).

The non-isothermal functions x^2-ay^2, y^2-ax^2, which are solutions of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}-\frac{\partial V}{x\cdot\partial x}-\frac{\partial V}{y\cdot\partial y}=0, \tag{35}$$

evidently satisfy the equation (31).

If α and λ are any two solutions of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + f'(x)\frac{\partial V}{\partial x} = 0, \tag{36}$$

where $f(x)$ is any given function of x, the condition (31) is satisfied and $\bar{\omega} = f(x)$.

If α and λ satisfy the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + a\frac{\partial V}{\partial x} + b\frac{\partial V}{\partial y} = 0, \tag{37}$$

$\bar{\omega}$ is of the form $ax + by + c$.

In general α and λ must both be solutions of an equation of the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + P\cdot\frac{\partial V}{\partial x} + Q\cdot\frac{\partial V}{\partial y} = 0, \tag{38}$$

where P and Q are any functions of x and y such that $\partial P/\partial y = \partial Q/\partial x$.

The question whether if (α, β) and (λ, μ) are orthogonal pairs and $(\alpha+\lambda, \beta + \mu)$ is not an orthogonal pair, it is possible to find a function (B) of β, and a function (M) of μ such that $(\alpha + \lambda, B + M)$ shall form an orthogonal pair, has already been answered; for α and λ must satisfy (31) in any case, and if they do this, $\bar{\omega}$ may be determined from (29) and (30) and B and M from (17) and (20).

VIII

ON THE DETERMINATION OF THE MAGNETIC BEHAVIOR OF THE FINELY DIVIDED CORE OF AN ELECTROMAGNET WHILE A STEADY CURRENT IS BEING ESTABLISHED IN THE EXCITING COIL [1]

More than fifty years ago Helmholtz established, on theoretical grounds, the now familiar equations for the manner of growth of a current in a circuit of constant inductance under a given electromotive force, and proved by a brilliant series of experiments [2] that the predictions of this theory were fulfilled in practice. It appeared, in particular, that if a circuit of resistance r containing a constant electromotive force, E, were closed at the origin of time, the current, I, would be given by the expression

$$\frac{E}{r}(1 - e^{-\frac{rt}{L}}), \tag{1}$$

if L were the "potential of the circuit upon itself," that is, the self-inductance. The "induced current" (i) would satisfy the equation

$$i = \frac{L}{r} \cdot \frac{dI}{dt} = \frac{E}{r} \cdot e^{-\frac{rt}{L}}. \tag{2}$$

If, therefore, I were plotted against the time, the resulting curve ($OCQKC$, Figure 1) would have as asymptote the straight line (ZC) parallel to the t axis at a distance E/r above it; the current in the circuit at any time (OP) would be given by the corresponding ordinate (PQ) of the curve and the instantaneous value of the induced current by the distance (NQ) at that time, of the curve from the asymptote. The whole "amount" of the induced current up to the given time would be represented by the shaded area (A) shut in by the curve, the asymptote, and the ordinates, $t = 0$, $t = OP$. If the electromo-

[1] *Proceedings of American Academy of Arts and Sciences*, xliii, no. 5, September, 1907.

[2] F. E. Neumann, *Abh. d. Berl. Akad.*, 1845 and 1847; Helmholtz, *Die Erhaltung der Kraft*, 1847; *Pogg. Ann.*, vol. lxxxiii, 1851; vol. xci, 1854; *Phil. Mag.*, vol. xlii, 1871.

tive force were suddenly shunted out of the circuit after the current had reached its final value, the "extra current" would have the value

$$\frac{E}{r} \cdot e^{-\frac{rt}{L}}. \tag{3}$$

Helmholtz also studied the "forms" of the currents induced in the secondary circuit of a small induction coil at the making and breaking of the primary circuit, and, by using in the apparatus iron cores, some of which were solid and some finely divided, he showed that the effect of eddy currents in the iron upon the apparent duration of the induced currents might be very appreciable. The results of Helmholtz's experiments were confirmed with the aid of other apparatus, during the next thirty years,[1] by a number of physicists.

The mathematical treatment of the subject begun by Neumann and Helmholtz was in 1854 pushed somewhat farther by Koosen, and in 1862 E. du Bois-Reymond [2] published an elaborate discussion of the equations laid down by Helmholtz for the determination of the currents in two neighboring circuits of constant self-inductances (L_1, L_2) and constant mutual inductance (M), and gave the solutions of the simultaneous equations

$$\begin{aligned} L_1 \cdot \frac{dI_1}{dt} + M \cdot \frac{dI_2}{dt} + r_1 I_1 &= E_1, \\ M \cdot \frac{dI_1}{dt} + L_2 \cdot \frac{dI_2}{dt} + r_2 I_2 &= E_2, \end{aligned} \tag{4}$$

corresponding to a number of different sets of physical conditions, in nearly the forms in which they now appear in textbooks. Du Bois-Reymond showed that if the secondary circuit contained no battery, and if, after the primary current had been fully established, its circuit were *suddenly* broken, the current induced in the secondary cir-

[1] Felici, *Ann. de Chimie*, vol. xxxiv, 1852; *N. Cimento*, vol. iii, 1856; vol. ix, 1859; vol. xii, 1874; vol. xiii, 1875. Cazin, *Compt. Rend.*, vol. lx, 1865; *Ann. de Chimie*, vol. xvii, 1869. Guillemin, *Compt. Rend.*, vol. l, 1860. Bertin, *Mem. de la Soc. des Sc. Nat. Strasbourg*, vol. vi, 1865. Bazzi and Corbianchi, *N. Cimento*, vol. iv, 1878. Bartolli, *Mem. d. Acc. d. Lincei*, vol. vi, 1882. Bazzi, *Att. d. Acc. d. Lincei*, vol. vi, 1882. Lemström, *Pogg. Ann.*, vol. cxlvii, 1872. V. Ettingshausen, *Pogg. Ann.*, vol. clix, 1876.

[2] Koosen, *Pogg. Ann.*, vol. xci, 1854. E. du Bois-Reymond, *Monatsberichte d. Berl. Akad.*, 1861, 1862. Brillouin, *Thèse*, 1880; *Jour. de Phys.*, vol. x, 1881; *Compt. Rend.*, 1882.

cuit would have a form like that of the dotted curve (P) in Figure 2; if after a few seconds the primary circuit were again closed, the secondary current when plotted against the time would yield a curve either like Q or like S in the same diagram. The lines in this familiar figure have been drawn to scale for a certain pair of circuits the self-inductances of which are equal, fixed quantities and the resistances

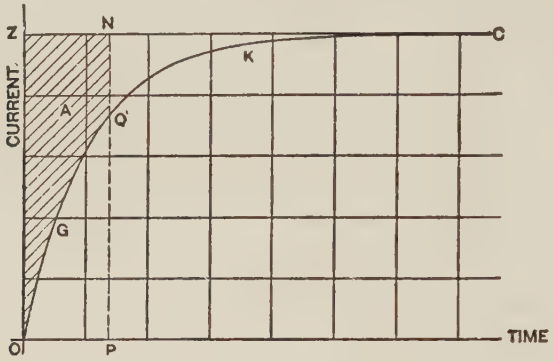


FIGURE 1. — If the current is expressed in absolute units (absamperes) and the time in seconds, the shaded area represents the change in the total flux of magnetic induction through the circuit, during the time OP.

also fixed. Q, R, S correspond to three different values of the mutual inductance (M), which are respectively half as great, nine tenths as great, and equal to the self-inductance (L) of either circuit. These curves show the currents induced in the secondary circuit when the primary is made; the crest of any such curve comes earlier the larger the value of M. The curve P, which represents a current induced in the secondary circuit when the primary circuit is broken, is drawn for the case $M = \frac{1}{2}L$, and therefore corresponds to the curve Q; E. du Bois-Reymond called attention to the fact that in such problems as this the areas V and W must be equal. The curves like P corresponding to R and S could be found merely by exaggerating all the ordinates of P in the ratio 9/5 or the ratio 2.

From the early days of induction coils, iron cores had been used to increase the mutual inductance of the circuits, and, soon after Helmholtz had given the equations for the currents in neighboring

circuits of constant inductances, coils containing iron were studied from the point of view of the principles which he had laid down. Helmholtz's own experiments and those of others soon showed, however, that the introduction of masses of magnetic metal into the space within the coils complicated very much their action. It appeared that

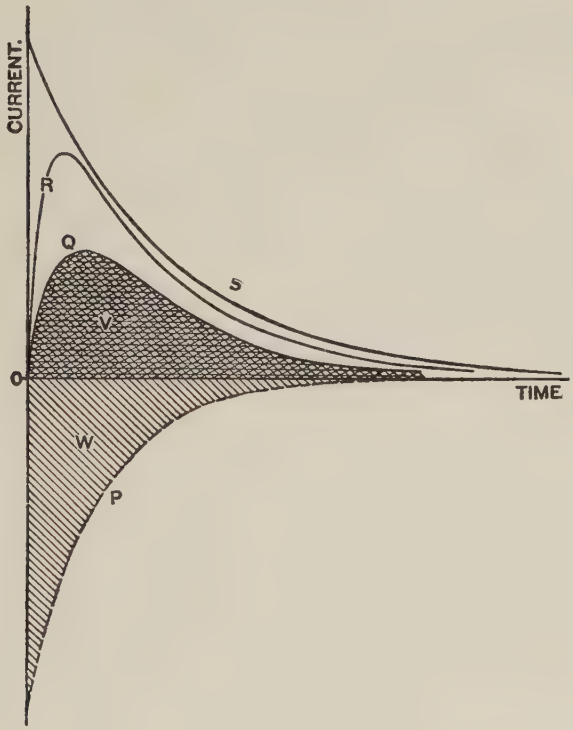


FIGURE 2. — The curves Q, R, S represent for different relative values of the mutual inductance the current induced in the secondary circuit of a certain induction coil without iron, when the primary circuit is suddenly closed.

the existence of eddy currents in the iron, if the coil were solid, and the fact that the counter electromotive force in a circuit — as measured by the time rate of change of the flux of magnetic induction through it — is by no means proportional to the rate of change of the intensity of the current if a circuit "contains iron," made the simple theory of Helmholtz inapplicable, as he himself had foreseen that

it would be. The subject interested many investigators,[1] who found it easy to exhibit the disturbing effects of eddy currents in hindering rapid magnetic changes in solid masses of iron and in thus modifying the characters of the induced currents; but it was not until much work had been done by many persons on the phenomena attending magnetic induction in iron that the theory of the alternate current transformer which had meanwhile come to be of much practical importance was very well understood. With the general introduction of dynamo-electric machinery the magnetic behavior of the different kinds of iron used in its manufacture became of practical interest, and several different magnetometric and ballistic methods of studying permeability were invented and employed in making the necessary measurements upon relatively small pieces of the metal.

Soon after the first hysteresis diagrams had been obtained as a result of experiments either on comparatively thin iron or steel rings, or on long, fine wires, it was found by engineers that, on account of the considerable time required to establish a steady current in the coil of a large electromagnet to which a given electromotive force had been applied, the "reversed current," and even the "step-by-step" ballistic methods which had proved effective in the cases of slender toroids, were, in their old forms at least, not well fitted for studying

[1] Faraday, *Researches*, 1831, 1832, 1846; Lenz, *Pogg. Ann.*, vol. xxxi, 1834; Henry, *American Journal of Science*, 1832; *Phil. Mag.*, vol. xvi, 1840; Dove, *Pogg. Ann.*, vol. xliii, 1838; vol. liv, 1841; vol. lvi, 1842; Beetz, *Pogg. Ann.*, vol. cii, 1857; vol. cv, 1858. Plücker, *Pogg. Ann.*, vol. lxxxvii, 1852; vol. xciv, 1855; Rayleigh, *Phil. Mag.*, vol. xxxviii, 1869; vol. xxxix, 1870; vol. xxiii, 1887; vol. xxii, 1886. Bichat, *Ann. de l'École Norm.*, vol. x, 1873; Sinsteden, *Pogg. Ann.*, vol. xcii, 1854; Magnus, *Pogg. Ann.*, vol. xxxviii, 1836; vol. xlviii, 1839; Schneebeli, *Bull. de la Soc. des Sc. Nat. de Neufchatel*, vol. xi, 1877; Blaserna, *Giorn. di Sc. Nat.*, vol. vi, 1870; Maxwell, *Electricity and Magnetism*, vol. ii, iv; Donati and Poloni, *N. Cimento*, vol. xiii, 1875; Stoletow, *Phil. Mag.*, vol. xlv, 1873; Auerbach, *Wied. Ann.*, vol. v, 1878; Rowland, *Phil. Mag.*, vol. xlvi, 1873; vol. xlviii, 1874; Thomson, *Phil. Trans.*, vol. clxv, 1875. J. Hopkinson, *Phil. Trans.*, vol. clxxvi, 1885; Von Waltenhofen, *Pogg. Ann.*, vol. cxx, 1863; Warburg, *Wied. Ann.*, vol. xiii, 1881. Wiedemann, *Lehre von der Elektricität;* Ewing, *Phil. Trans.*, vol. clxxvi, 1885; *Proc. Royal Soc.*, 1882, "Magnetic Induction in Iron and other Metals"; Du Bois, *The Magnetic Circuit;* Fleming, *The Alternate Current Transformer;* Ewing and Low, *Proc. Royal Soc.*, vol. xlii, 1887; *Phil. Trans.*, vol. clxxx, 1889; Du Bois, *Phil. Mag.*, 1890; Oberbeck, *Wied. Ann.*, vol. xxii, 1884; J. and E. Hopkinson, *Phil. Trans.*, vol. clxxvii, 1886; Jouaust, *Compt. Rend.*, vol. cxxxix, 1904; E. Hopkinson, *Brit. Assoc. Report*, 1887; Tanakadaté, *Phil. Mag.*, 1889; Wilson, *Proc. Royal Soc.*, vol. lxii, 1898. Baily, *Phil. Trans.*, vol. clxxxvii, 1896. Many other references may be found in these sources.

the magnetic properties of such massive closed iron circuits as frequently occurred in practice. When there was a gap in such a circuit, the problem, of course, offered no difficulty, but when large iron frames were completely closed, it became the custom, in satisfying commercial contracts, to attempt to get information about the permeability of the metal as a whole from tests, under given conditions, upon small, thin specimen pieces made as nearly as possible of the same material as the original, or else cut from it. It was usually impossible, however, to be sure that the temper of the small piece was sufficiently like that of the mass to make it a fair representative of the whole, and the preparation of the specimens was often troublesome, so that some more practical method of procedure was seen to be desirable,[1] and it seems to have occurred to a number of different persons independently that a good deal might be learned about the magnetic properties of the core of an electromagnet if one determined the manner of growth of a current in an exciting coil of a given number of turns wound closely about the core, when, under given initial conditions, a constant, known, electromotive force was applied to the coil circuit.

The Determination of some of the Magnetic Properties of the Core of an Electromagnet from the March of a Current in the Exciting Coil

If, at any instant, the total flux of magnetic induction through the n turns of the exciting coil of an electromagnet is N (maxwells), if r is the resistance of the coil circuit (in ohms), i the current in it (in amperes), and E the applied electromotive force (in volts), then

$$E - \frac{1}{10^8} \cdot \frac{dN}{dt} = ri, \tag{5}$$

or

$$\frac{dN}{dt} = 10^8 \cdot r\left(\frac{E}{r} - i\right); \tag{6}$$

and if the final value (E/r) of the current be denoted by i_∞ and the change in N during the time interval t_1 to t_2 by $N_{1,2}$,

$$N_{1,2} = r \cdot 10^8 \cdot \int_{t_1}^{t_2} (i_\infty - i)\, dt. \tag{7}$$

[1] Drysdale, *Jour. Inst. Elec. Engineers*, vol. xxxi, 1901.

If, now, i be plotted against the time in a curve s (Figure 3) in which l centimeters parallel to the axis of abscissas represent one second, and an ordinate m centimeters long one ampere, the curve will have an asymptote, CY, parallel to the axis of abscissas, at a distance, KC, from it corresponding to E/r amperes, and, if OK represents the time

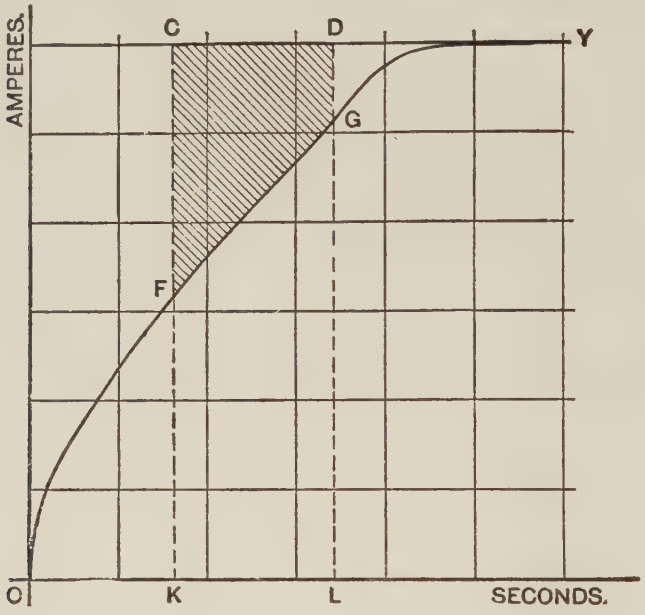


FIGURE 3. — If l centimeters parallel to the horizontal axis represent one second, and an ordinate m centimeters long one ampere, $A \cdot 10^8 \cdot r/lm$ (where A is the area, in square centimeters, of $CDGF$) represents the change in the magnetic flux through the circuit during the interval KL.

t_1, and OL the time t_2, the area $FGDC$, or $A_{1,2}$, expressed in square centimeters, is equal to

$$lm \int_{t_1}^{t_2} (i_\infty - i)\, dt, \tag{8}$$

so that

$$N_{1,2} = \frac{r \cdot 10^8 \cdot A_{1,2}}{lm} = \frac{10^8 \cdot E \cdot A_{1,2}}{lm \cdot i_\infty}. \tag{9}$$

In practice N usually differs from $n\phi$, where ϕ is the induction flux through the iron core of the electromagnet alone, by only a small frac-

tion of itself, and, if a is the area of the cross section of the core at any point, a certain average value of B, the induction, can be obtained from the expression N/na, though in such cores as are used in large transformers, H, and therefore B, would probably have very different values at different points of the section. Really N is greater than $n\phi$ by the amount of the magnetic flux, in the air about the core, through the turns of the exciting coil, caused by the current in the coil itself or by neighboring currents, if there are such.

Using this theory, a good many persons have studied at various times the magnetic properties of different large masses of iron, and in 1893 Professor Thomas Gray of Terre Haute published in the Philosophical Transactions of the Royal Society a long series of very beautiful current curves,[1] obtained, with simple apparatus handled with great skill, from a 40 K. W. transformer belonging to the Rose Polytechnic Institute. A number of diagrams[2] showing the manner of growth of currents in the exciting coils of large electromagnets with solid cores have been printed within the last dozen years; of these the curves given by Dr. W. M. Thornton are especially interesting.

If to the coil of an electromagnet, in series with a rheostat of resistance r, a given electromotive force be applied, and if r be then reduced by steps, at intervals so long that one is sure that the final current belonging to each stage has been practically attained, the curve which has elapsed times for abscissas and the corresponding values of the strength of the current for ordinates, will have the general form of the line U in Figure 4, though, if the core be so large that the building-up time at each stage is long, the diagram will be much drawn out horizontally. The curve which shows the march of the current when the electromotive force is applied directly to the coil without the intervention of the rheostat will resemble line V in the same figure. The exact forms of these curves depend, of course, upon the magnetic state of the core at the outset, and will be very different if the iron has been thoroughly demagnetized before the observation

[1] T. Gray, *Phil. Trans.*, vol. clxxxiv, 1893.

[2] Hopkinson and Wilson, *Journal of the Institute of Electrical Engineers*, vol. xxiv, 1895; Thornton, *Electrical Engineer*, vol. xxix, 1902; *Phil. Mag.*, vol. viii, 1904; *Electrician*, 1903; Peirce, *Proceedings of the American Academy*, vol. xli, 1906. Several figures from this last paper are here reproduced.

is made, or if it be strongly magnetized. Figure 5,[1] which illustrates this fact for some V curves, records some measurements made upon a 15 K. W. transformer (R) belonging to the Lawrence Scientific School. In the case represented by each line the core was previously magnetized in one direction with the full strength of the current, and the circuit was then broken and left open for a few seconds. With the electromotive force in it unchanged in intensity, but in some instances changed in direction, the circuit was then closed again and a current curve obtained. If the electromotive force has its old direction, such

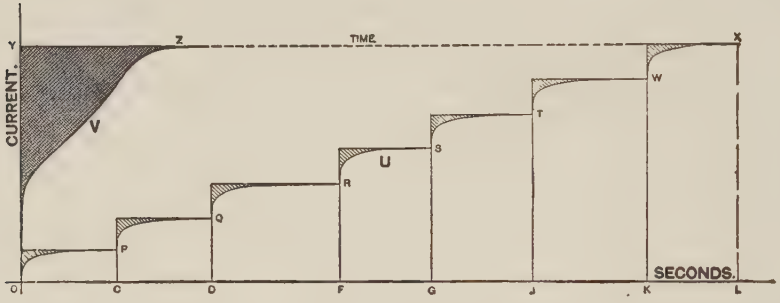


FIGURE 4. — Curves which represent the growth of the current in the exciting coil of an electromagnet when (V), the circuit which has the resistance r, is closed and left to itself; and when (U), the circuit, is closed when it has a comparatively large resistance, which is then reduced to r by steps.

a curve is said to be "direct"; if the new direction is the opposite of the old, the curve is called "reverse." In one case the magnetic journey of the core during the rise of the current is represented approximately by the portion PFM of the corresponding hysteresis diagram (Figure 6); in the other case the journey follows the arc $QUZM$. Lines 1, 2, and 4 in Figure 5[1] are reverse lines, while 3 and 5 are direct.

In Figure 4 the line OY corresponds to the final value (i_∞) of the current, and if its length in centimeters is mi_∞ and if A is the area in square centimeters shut in by OY, YX, and V, it is evident that in the case represented by V the whole change in induction flux through the turns of the coil due to the current is

$$\frac{10^8 \cdot E \cdot A}{l \cdot OY}.$$

[1] Identical with Fig. 11 on p. 73.

In the case represented by the line U, $(10^8 \cdot E/l)$ times the sum of the terms formed by dividing each of the small shaded areas by the ordinate, expressed in centimeters, of its upper straight boundary, gives the change in the induction flux through the turns of the coil due to the current when it grows in the manner indicated. Of course if the current is not allowed time to attain its final value at each stage, a serious error may be introduced.

The amount of flux which, in a given large mass of iron, in a given magnetic condition at the outset, corresponds to a current of given final strength in the exciting coil, usually depends in some slight degree upon the manner of growth of the current. If after a large core has been magnetized in one direction by the steady application of a given electromotive force until the current has reached its full value, the exciting circuit be broken, and, after the direction of the electromotive force has been reversed, closed again, it sometimes happens that the magnetic flux after the new current has attained its maximum value is slightly less when the current follows the course of the curve V than when it grows by short stages in the manner indicated by the curve U. If, however, there are but two or three steps, the difference is, as a rule, of no practical importance, and if one has a suitable oscillograph or other recording instrument, it is possible to get a set of current curves for any given maximum value of the current from which an extremely good statical hysteresis diagram may be obtained for the core.

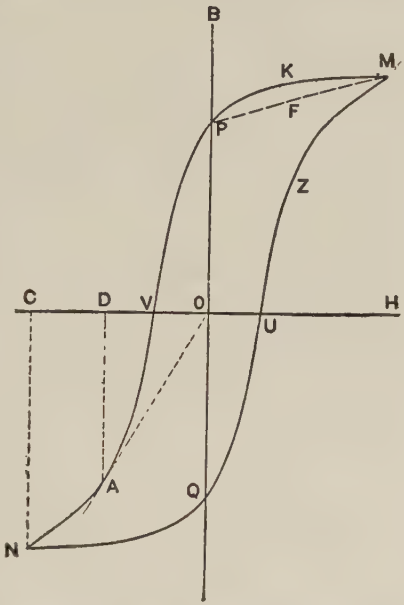


FIGURE 6.—When a direct current curve is taken, the core of the electromagnet makes a magnetic journey represented approximately by the arc PFM; in the case of a reverse curve the core follows the line $QUZM$.

If while a steady current from a constant storage battery of voltage E is passing through the coil of an electromagnet, the resistance of the

coil circuit be suddenly increased to a new value r_1, so that the current (i) will ultimately fall to a lower value represented by ON in Figure 7, the current curve, which has been a horizontal line, sinks in such a manner as to become asymptotic to the horizontal line NB. At any instant after the change,

$$E - \frac{dN}{dt} = r_1 i, \tag{10}$$

in absolute units, so that in volts, ohms, amperes, and maxwells,

$$N_{0,1} = 10^8 \int_{t_0}^{t_1} (E - r_1 i)\, dt = 10^8 \cdot r_1 \cdot \int_{t_0}^{t_1} (i_1 - i)\, dt. \tag{11}$$

If an abscissa l centimeters long corresponds to one second, and an ordinate m centimeters represents one ampere, and if $A_{0,1}$ stands for

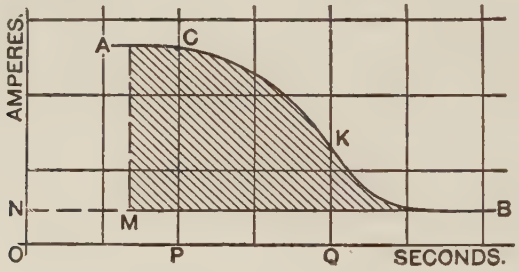


FIGURE 7.—The shaded area represents on a certain scale the change in the flux of magnetic induction through a circuit when the resistance of the circuit is suddenly increased and then kept constant.

the area in square centimeters bounded by the current curve, the asymptote, and ordinates corresponding to the times t_0, t_1, the change in the flux of magnetic induction through the circuit during this time-interval is (in maxwells)

$$\frac{10^8 \cdot r_1 \cdot A_{0,1}}{lm}. \tag{12}$$

If, after a current has been built up by stages in the coil of an electromagnet, in the manner indicated by curve U of Figure 4, the process be reversed, and the resistance of the circuit be increased by steps, the current curve will look very much as the curve U would if looked at from the wrong side of the paper when upside down.

As has already been stated, it is possible to get slightly different hysteresis diagrams for a massive core originally demagnetized, when the current is made to change from a given positive limit to the negative limit in different ways; and it is important, in predicting the behavior of a magnet which is to be used for a given purpose, to employ in computation the hysteresis diagram which corresponds to the particular magnetic journey which the core will take in practice. A

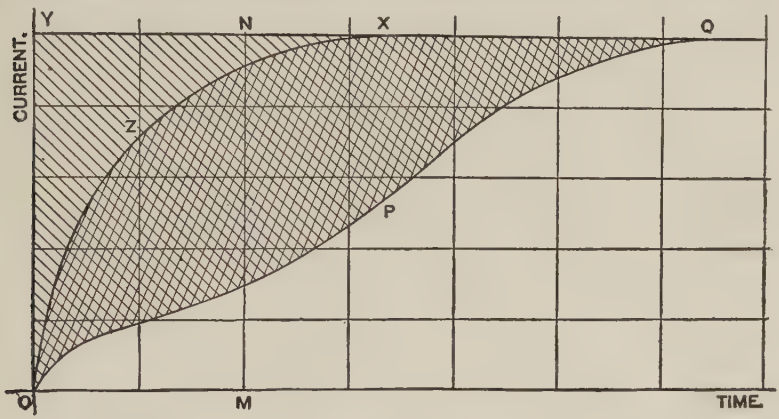


FIGURE 8. — The areas between the asymptote and the curves Z and P are proportional to the changes of magnetic flux through the circuit caused by direct and reverse currents of the same final strength.

single carefully made curve of the U type with a dozen steps will, however, give a result good enough for any commercial purpose, though my own experience shows that it is not always easy to measure all the small areas, especially the lower ones, with the desirable accuracy, when the width (OY) of the whole diagram is only 12 or 14 centimeters.

If in the U diagram there is only one intermediate stage, and if the core is in a given magnetic condition at the outset, the change in the magnetic flux, due to a current of given final value, ought not to differ by more than perhaps a fraction of one per cent from the corresponding change when there is no intermediate step and the case is represented by V. Sometimes a series of U diagrams, each with but one intermediate step, at a place determined by a proper choice of r, may be made to yield very accurate information about the permeability of the large mass of metal which will suit some special use of the magnet.

Figure 8, which resembles in general design some diagrams given by Dr. Thornton, shows a "direct curve" (Z) and a "reverse curve" (P) for a certain magnet. The area $OZXY$ represents the change of magnetic induction when the core covers the arc PFM (Figure 6) on the hysteresis diagram belonging to the journey; the area $OPQXY$ represents the change of magnetic flux when the core takes the journey corresponding to the arc $QUZM$ on the hysteresis diagram. The doubly shaded area represents the flux change corresponding to the line $QUZMKP$.

The Uses of Exploring Coils wound upon the Core of an Electromagnet

If an electromagnet, in addition to its exciting coil, has another wound about its core, and if the observer has means of obtaining the intensity (i') of the current induced in this secondary coil, for given current changes in the exciting coil, as a function of the time, it is easy to study the magnetic properties of the core under the circumstances of the experiment. Let there be n' turns in the secondary coil, let the resistance of its circuit be r' ohms, and let N' be the total induction flux, in maxwells, through the turns of the coil at the time t, then if i' is measured in amperes

$$\frac{dN'}{dt} = -10^8 \cdot r' \cdot i'. \tag{13}$$

If i' be plotted against the time in a curve in which l' centimeters parallel to the axis of abscissas represent one second and an ordinate m' centimeters long one ampere, and if $A'_{1,2}$ represents the area between the curve, the axis of abscissas and the ordinates corresponding to the time t_1, and t_2, we have in absolute value,

$$N_2' - N_1' = 10^8 \cdot r' \int_{t_1}^{t_2} i' \cdot dt = \frac{10^8 \cdot r' \cdot A'_{1,2}}{l'm'} = q' \cdot A'_{1,2}, \tag{14}$$

where q' is a known constant.

When the primary current (i) in the exciting coil is growing, the current in the secondary coil has a direction opposite to that of i, and it is often desirable to emphasize this fact in a diagram by drawing

the i, t and i', t curves on opposite sides of the axis of abscissas; but if the relative values of i and i' are alone to be considered, it is sometimes more convenient to disregard their relative directions. If in any case the current in the exciting coil of an electromagnet be made to grow in the manner indicated by curve U in Figure 4, the i', t diagram will consist (Figure 9) of a set of detached areas on the t axis. The sum of any number of these areas when multiplied by $10^8\, r'/l'\, m'\, n'$ gives approximately the whole change in the induction flux through the core up to the corresponding time, from the outset. In the "step-by-step" ballistic method of determining the permeability of a closed ring of rather small cross-section the areas represented by the shaded

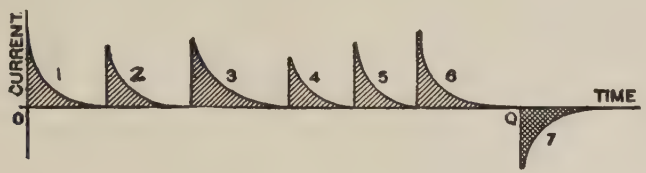


FIGURE 9. — A portion of the record of an oscillograph in the circuit of a secondary coil wound on the core of an electromagnet when the current in the exciting coil is made to change by sudden steps in the determination of a hysteresis cycle.

portions of Figure 9 are determined by discharging the induced current through a calibrated ballistic galvanometer of long period, and assuming that the first elongations of the suspended system measure these areas directly. As will appear in the sequel, it is possible, though not very easy, to get good results in this way, even if the cross-section of the laminated core is as great as, say, 800 square centimeters; for this, however, a properly constructed galvanometer is required.

The "time constant" of a circuit in which a current of given final intensity is to be established is shorter the higher the electromotive force used to generate the current; it is desirable, therefore, to employ a battery of rather high voltage and to reduce the current by non-inductively wound resistance in series with the exciting coil of the electromagnet. If a moving coil galvanometer is used, it is often necessary to correct for the effect of the counter electromotive force induced in the coil as it swings in the field of its own permanent magnet, and it is always necessary to use steps so short and to make the period

of the galvanometer so long (perhaps 300 or 500 seconds) that the practical duration of the induced current may be small in comparison. It is usual to send the current to the exciting coil by means of a commutator and a long series of manganine resistance coils capable of carrying the desired currents; these coils are often mounted in a frame furnished with some device by which any or all of them can be shunted out of the circuit at pleasure. Two rheostats, made for this purpose some years ago by the Simplex Electric Company, have been found by the staff of the Jefferson Physical Laboratory very satisfactory in practice. By means of such a set of coils as those just described, one may easily get either a progressive, step-by-step increase or decrease in the current, or a reiteration of any particular step. One convenient way of arranging the apparatus for the repetition at pleasure of any desired step has been recently described by A. H. Taylor.[1] The method of reversals is usually unsatisfactory with large cores. A set of adjustable electrolytic resistances fitted for carrying heavy currents is often useful.

In the case of a very large closed electromagnet, even if the core be laminated, it is extremely difficult to get very useful results by aid of a ballistic galvanometer of short period, but if one has a suitable oscillograph or other recording instrument at hand, it is easy to obtain a diagram something like that shown in part in Figure 9, though it is necessary to make sure that the intervals between the steps, unlike those in this figure, are long enough to record the whole of each induced current.

If the primary current (i, t) curves are to be used in studying the magnetic changes in the core of an electromagnet, the sensitiveness of the oscillograph must be so adjusted that the deflection due to the largest value of the current (U, Figure 4) will make a record on the paper; if the (i', t) curves are to be used, the steps may be as numerous as one likes, and the sensitiveness of the recording instrument may be so great that, starting from the base line, the record of the highest induced current shall just fall on the drum. In this latter case the areas to be measured may be made so large that any uncertainty as to

[1] A. Hoyt Taylor, *Phys. Rev.*, vol. xxiii, 1906; Mordey and Hansard, *Elect. Engineer*, vol. xxxiv, 1904; Searle and Bedford, *Phil. Trans.*, vol. cxcviii, 1902; Drysdale, *Jour. Inst. Elect. Engineers*, vol. xxxi, 1901; Lamb and Walker, *Electrical Review*, vol. xlviii, 1901.

the exact time when any induced current may be considered to end is unimportant. When many records are taken on the same paper, the drum has an opportunity to revolve a good many times during the operation, and it is not always easy to decipher the complicated maze of curves. Of course the fact that an electromagnet has a closed secondary circuit modifies somewhat the form of the building-up curve in the primary, but, theoretically at least, this should not affect the value of the magnetic flux due to the primary current if its final intensity is given, and the difference is inappreciable if there are only a few turns in the secondary coil.

Instead of changing the resistance in the primary circuit suddenly, at each step, Dr. Thornton, in dealing with the frames of some very large dynamos, made each step gradually, by moving an electrode slowly in a trough of acidulated water from one stopping place to another. Figure 10 is a close copy of one of his records published in the "Philosophical Magazine" for 1904.

Fluxmeters and Quantometers

Given an amperemeter of the ordinary d'Arsonval type, in which an open-frame, low resistance, unshunted coil swings in the strong magnetic field between an interior soft iron core and the hollowed-out jaws of a powerful magnet, it is often possible to make the controlling springs so weak that if the coil circuit be suddenly closed on itself while the coil is in motion, the damping effects of the induced currents will bring the coil almost instantly to rest wherever it may happen to be, and, until the circuit is broken, the coil will keep its position fairly well. Several years ago Dr. R. Beattie [1] showed that if the ends of a low resistance exploring coil (A) be electrically connected with an instrument of this kind, and if the flux of magnetic induction through A be changed during the time interval T by an amount N, the coil will move from its initial position to a new position through an angle proportional to N and, apart from pivot friction, practically independent, within wide limits, of T.

The "quantometer" first made by Dr. Beattie had a coil of twenty-four and a half turns wound on a metal frame and suspended on a

[1] R. Beattie, *Electrician*, Dec. 1902.

single needle point between the poles of an electromagnet; the ends of the coil dipped into mercury cups fixed to the case of the instrument. In the kind of fluxmeter now common, the coil is hung by a silk fibre (or a quartz thread) from a spring, so as to avoid pivot friction; a permanent magnet is used, and the current is led into and out of the coil through helices of very fine silver or copper gimp; the resistance of this gimp is sometimes much greater than that of the coil itself, and for laboratory use it is often well to employ mercury cups, as Dr. Beattie did, so arranged as to minimize the disturbing effects of capillarity. The original quantometer had a resistance of only one ohm.

Many persons who have attempted to use very strong electromagnetic fields in d'Arsonval galvanometers have found that it is very

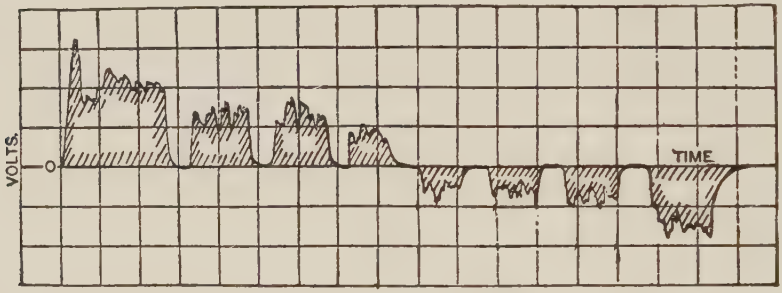


FIGURE 10. — Typical record for half a hysteresis loop, given by Dr. Thornton.

difficult to procure insulated copper or silver wire for the suspended coil so free from paramagnetic properties that the coil shall not have a permanent "set" in the field, too strong to be conveniently controlled by the torsion of the gimp through which the current enters the coil. In the case of a quantometer where there is practically no controlling moment from the suspending fibre, the paramagnetic properties of the coil may be very troublesome; and in some of the most recent instruments the angular movements of the coils, due to given changes of induction through the turns of the exploring coils, are somewhat different according as the movement is towards the left or towards the right. If a telescope and scale be set up in such a position that the behavior of the coil can be watched after it has moved through a considerable angle, urged by a sudden, definite change of flux in the exploring coil, it will often be found that the coil does not remain even

approximately at rest, but moves steadily and so rapidly that a considerable error is introduced if the given change of flux through the exploring coil is made slowly. It is desirable, therefore, to test an instrument of this kind carefully before using it.

If great accuracy is not required, a good fluxmeter, of some standard make, and of sensitiveness suited to the work to be done, is, in experienced hands, a most useful instrument; the time needed to establish a current of given strength in the coil of a large electromagnet with a solid core may be several minutes, but a very good fluxmeter will, nevertheless, show directly, with an error of not more than 2 per cent, the change of magnetic flux through the core.

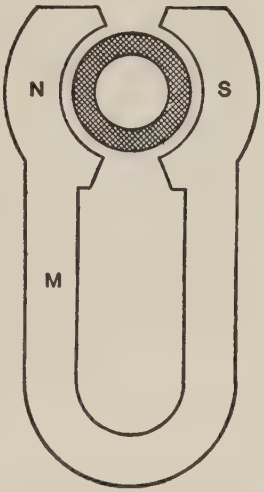


FIGURE 11. — Plan of one of the permanent magnets of the fluxmeter *F*; the shaded area represents the cross-section of the soft iron core.

If the fluxmeter coil is not wound on a closed metal frame, the mutual damping effect of currents in the coil and in the core which it surrounds are not always effective unless the resistance of the external circuit, made up of the exploring coil and its leads, is fairly small compared with the resistance of the suspended coil itself. An instrument, therefore, which works very well with an exploring coil of a small number of turns often becomes quite useless when, in order to get the required sensitiveness, the observer tries to employ an exploring coil made of many turns of fine wire. On the other hand, if a fluxmeter of this kind is too sensitive for a given piece of work, it is not always easy to reduce the sensitiveness quickly.

If the flux changes to be measured are large, it is often convenient to have a fluxmeter the coil of which consists of a few turns either wound on a copper frame or else accompanied by several turns of stout wire closed on themselves. It is possible to use such an instrument with many different exploring coils and to change its sensitiveness within wide limits by varying the resistance of the external circuit.

In doing a small part of the work described below, I was able to use either a Grassot Portable Fluxmeter, or a certain fixed laboratory

fluxmeter (F) furnished with a tall chimney to hold the 140 centimeter long fibre by which the coil was suspended. The cast-iron magnet of this last-mentioned instrument had, when finished, the form shown in plan in Figure 11 and was 45 mms. thick. The casting was made with a web connecting the poles, and this was removed after the hole for the coil had been cut out and finally reamed to a diameter of exactly 5 cms. on a Browne and Sharpe milling machine. The magnet was hardened and treated by Mr. G. W. Thompson, the mechanician of the Jefferson Physical Laboratory, who has had much experience in this kind of work. During the process the poles were held in position by an iron yoke. The core (shaded in the diagram) within the coil is 41.3 mms. in outer diameter, and is about 7 mms. thick. The instrument was constructed and set up by Mr. John Coulson, who has helped me in countless ways during the progress of the work. It was comparatively easy to substitute one of the set of coils belonging to this fluxmeter for another. For certain purposes it was convenient to have a coil of 200 turns of stout insulated wire which was wound about the magnet, though the latter had a large permanent moment.

The Coefficients of Self-Induction of a Circuit which has an Iron Core

When many years ago it was found that the induction B at a given point in a piece of iron exposed to a given magnetic field H is not only not in general proportional to the intensity of the exciting force, but is not even determined when H is given, it became evident that no such constant can exist in the case of an inductive circuit which "contains" a magnetic metal as was assumed in the conception of Neumann's "Electrodynamisches Potential," [1] and that the different common definitions of self-induction, when applied to an electromagnet of the usual form, really describe physical quantities which are widely different from one another. The ambiguity in the use of the term "self-induction" still exists, and it will be convenient in this paper to adopt the notation used by Sumpner [2] in his article on "The Variations of the Coefficients of Induction." If, in absolute value, I is the strength of a current growing in the coil of an electromagnet with

[1] Neumann, *Abh. d. Berl. Akad.*, 1845.
[2] Sumpner, *Phil. Mag.*, vol. xxv, 1888.

laminated core, if N is the total flux of magnetic induction through the turns of the coil, and e the counter electromotive force of induction, we may call the ratio of e to the time rate of change of the current, L_1, the ratio of N to the current, L_2, and the ratio, to I^2, of twice the contribution (T) made by the current to the energy when there are no other currents in the neighborhood, L_3, so that

$$\begin{aligned} e &= L_1 \cdot \frac{dI}{dt}, \qquad N = L_2 \cdot I, \qquad L_1 = \frac{dN}{dI} \\ e &= \frac{d(L_2 \cdot I)}{dt}, \qquad T = \tfrac{1}{2} L_3 \cdot I_2, \quad L_1 = L_2 + I \cdot \frac{dL_2}{dI}. \end{aligned} \tag{15}$$

If then for a particular magnetic journey, taken at a given speed, N is given as a function of I in the form of a curve like OPQ in Figure 12, the value, at any point P on the curve, of L_1 is the slope of the curve or the tangent of the angle XKP; the value of L_2 at P is the slope of the line OP or the tangent of the angle XOP; the value of L_3 is the ratio of twice the curvilinear area OPD to the area of the square erected on OJ. Similar definitions are sometimes given for such a magnetic journey as is represented by the line $MGPQ$ of Figure 13.

In the paper just cited Sumpner gives a very interesting graphical method of constructing a curve which shall show the manner of growth of the current in the coil of the electromagnet when the curve which connects N and I is given.

The Electromagnets used in doing the Work described below

A number of electromagnets were used in carrying on the experimental work described in this paper.

Though the investigation had to do primarily with magnets the cores of which were laminated or otherwise finely divided so as to get rid in great measure of the disturbing effects of eddy currents, one or two large magnets with massive cores were useful for purposes of comparison. One of these (P), which weighs about 1500 kilograms, has the general shape shown in Figure 14.[1] The outside dimensions of the frame proper are about 101 cms. × 80 cms. × 40 cms. The base is

[1] Identical with Fig. 1 on p. 51.

of cast iron and of rectangular cross-section (20 cms. × 40 cms.), the cylindrical arms are of soft steel 25 cms. in diameter, the rectangular pole pieces are 4.5 cms. thick, and the area of each of the opposed faces is about 580 square centimeters. The four coils have together 2823 turns, and a resistance at 20° C. of about 12.4 ohms.

Figure 15 shows in outline the electromagnet Q, which weighs about 300 kilograms: the core has a square cross-section of about 156 square centimeters area, and is built up, cobhouse-fashion, of soft iron plates about one third of a millimeter thick, each of which was immersed in thin shellac and then thoroughly baked in an electric oven before it

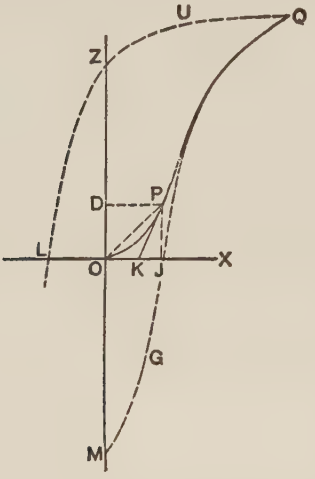


FIGURE 12. — This illustrates different meanings of the word *inductance*.

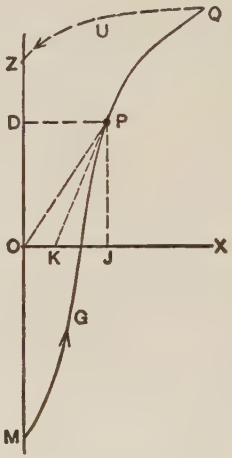


FIGURE 13

was used. Each of the spools, which are practically alike, weighs about 30 kilograms and has four coils, an inner one forming a single layer, the next forming three layers, and the two outer ones wound together side by side from two supply spools, and each equivalent to five layers; in all, both spools together have 3883 turns. The whole core frame is about 74 cms. long and 62 cms. broad. One stratum 2.5 cms. high and reaching across the middle of the core (Figure 16*a*) within one of the spools, is made up of five portions insulated from one another, and each of these is surrounded by an exploring coil of insulated wire.

Figure 16*b* shows the form of the cross-section of the rectangular core frame of a 15 kilowatt transformer (*R*) constructed for experimental purposes and belonging to the Lawrence Scientific School. Besides a low-resistance primary coil, this transformer has 19 similar coils each of about 85 turns, any number of which may be connected to form a secondary circuit. The outside dimensions of the core frame are about 78 cms. and 34 cms.; the area of the cross-section of the finely divided core is about 108 square centimeters.

Magnet *S* has a core consisting of two round solid pieces 76 cms. long and 7.4 cms. in diameter with axes 24 cms. apart, connected together at the ends (so as to form a rectangular frame) by two massive iron blocks. This magnet has two spools, each of which has two coils formed by winding two strands side by side; the whole number of turns is 1724.

The core of magnet *T* forms a square 58 cms. long on the outside and 53.5 cms. wide. Its cross-section is a rectangle 7.5 cms. by 6.7 cms. The core is built up of sheet metal 0.38 of a millimeter thick.

Through the kindness of Dr. George Ashley Campbell I have been allowed to use also seven toroidal coils (of inductances between 0.3 and 13 henries) wound on cores made of very fine (No. 38 B. & S.) iron wire. Such cores are, of course, extremely expensive, but the disturbing effects of eddy currents in them are practically negligible for the purposes of this paper.

The Demagnetizing of the Core of a Large Electromagnet

In order to be able to study satisfactorily the magnetic properties of a given piece of iron or steel, it is usually necessary that one should know with some accuracy the magnetic state of the specimen at the outset, and, especially when the metal has the form of a closed ring or frame, the previous history of which is unknown, the only safe procedure is to demagnetize the iron as completely as possible before one makes any experiments upon it. If the metal has the form of a long rod in a solenoid, or of a slender ring wound about uniformly with insulated wire and magnetized in the direction of its circumference, it is easy to send through the coil which surrounds the iron a long series of currents alternately in opposite directions, which, starting with a value that shall subject the core to a magnetic field at least as strong

as any to which it has been previously exposed, gradually decrease in intensity to zero. One common way of doing this is to attach the coil to the secondary of a sufficiently powerful alternate current transformer so arranged that the primary coil may be slowly withdrawn to a long distance from the secondary. In the case of the soft iron wire

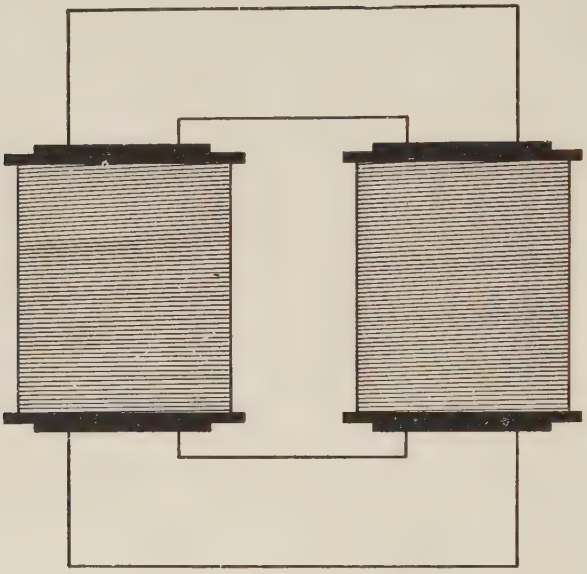
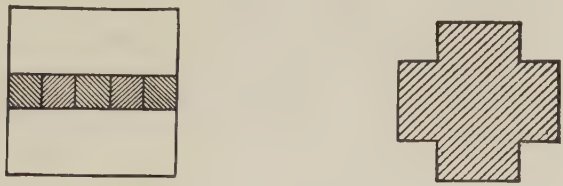


FIGURE 15. — The electromagnet Q, which has a laminated core made of sheet iron one third of a millimeter thick and weighs about 300 kilograms.

the demagnetization is sometimes accomplished by heating the wire red hot.

It is often a matter of considerable difficulty to remove entirely the effects of previous magnetization from the completely closed massive core of a large transformer: even if the source of a current in the exciting coil has a high voltage, several seconds may be required to establish the current, and the use of an alternating demagnetizing current in the coil, with any commercial frequency, is barred out. If a powerful storage battery be connected to the exciting coil through a commutator and a suitable "liquid rheostat," one may begin with a sufficiently strong current (I_0) and, after reversing this several times by hand, increase a little the rheostat resistance so as to decrease the

current slightly, then reverse this weaker current a number of times, and thus proceed until the current is reduced to a very small value; but if the core is very large, the operation may take a couple of hours even if the number of steps is not excessive, and after all, it is not easy to tell whether the work has been successful. If the initial current was strong enough, if the stages were sufficiently numerous and properly spaced, and if the number of reversals at each step was great, one may, of course, expect to find the core pretty thoroughly demagnetized, but to test the matter it is usually necessary to undo what has



a *b*

FIGURE 16. — Forms of the cross-sections of the laminated cores of the electromagnets Q and R.

been accomplished by determining the amount of magnetic flux sent through the core when a current of given intensity (I) is sent through the exciting coil. This amount ought to be the same whether this testing current has the same direction as that of the last application of the large current (I_0) or the opposite direction, and unless one has a hysteresis diagram for the core obtained by using currents which range exactly between $+ I_0$ and $- I_0$ the whole work must be done twice. The determination of the flux changes may be made very conveniently with the help of a fluxmeter, but if the highest accuracy is required, it is better to take an oscillogram of the building-up curves of the current when the core starts from its state of supposed neutrality.

If the core of a large electromagnet is not quite closed, it is comparatively easy to demagnetize the iron almost completely and to prove that this has been done; indeed, if the gap has the proper width, the iron practically demagnetizes itself in a wonderful manner. An instance of this was given by Professor Thomas Gray in the case of a 40 K. W. transformer, and I found that the hysteresis diagram for a certain electromagnet which has a solid core the area of which in its slenderest part is more than 450 square centimeters, consists

practically of a single straight line when the air gap has a width of 35 millimeters. With this magnet, using an excitation of either 7800 ampere-turns or 15,800 ampere-turns, I obtained current-time curves which were wholly indistinguishable even when much enlarged and superposed on a screen, whether the current had the same direction as its predecessor or the opposite direction.

If the core of an electromagnet happens to be a straight bar, or a straight bundle of wire, it may be demagnetized by a long series of

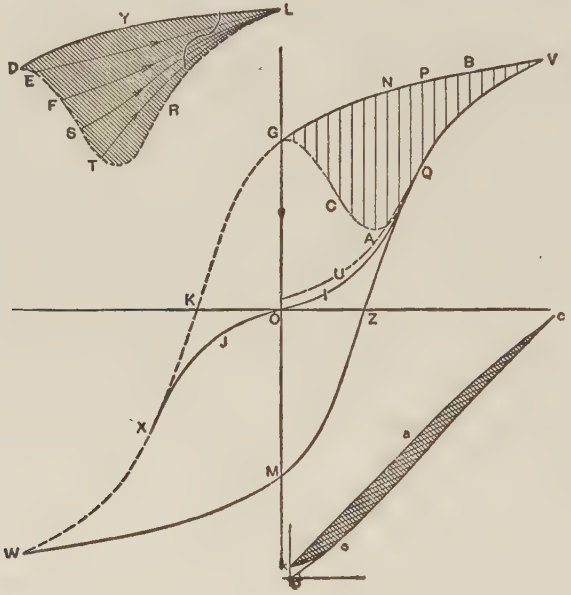


FIGURE 17

currents which have alternately one direction and the other, and which slowly decrease in intensity from an initial value which may be considerably smaller than the current which magnetized the iron. Figure 17 shows the results of experiments upon a rod of soft steel 80 diameters long in a long solenoid. The arrangement of the apparatus is shown in Figure 18.[1] The extreme value of the magnetizing field was 27 gausses, and the average moment per cubic centimeter which the field caused was 246. At the outset the core was thoroughly demagnetized, then a series of steady currents, each a little stronger than the

[1] Identical with Fig. 1 on p. 89.

last, was sent through the coil, and the moment of the rod was determined for each direction of the current. This gave the curve *WXOQV*. Then the hysteresis diagram *VGKWMZV* was obtained, and after the core had returned to the condition indicated by the point *V*, the current was somewhat decreased until the core "reached" the point *B*, and then this current was reversed in direction one hundred times, after which (when the current had the positive direction) the iron had exactly arrived at the point on the curve *OIQV* beneath *B*. The core was then brought to *V* again, the current was decreased, — this time until the core reached the point *P*, — this current was reversed one hundred times, and it was then found that when it ran in positive direction the core had arrived at the point *Q*. This process, repeated for many points on the line *GPV*, yielded the curve *VQACG*. If after being at *V* the core was brought to a point between *P* and *N*, and if after it had been many times reversed the current was decreased by short steps with many reversals at each stage, the core traversed the curve *U*, whereas if the first drop carried the core no farther than *P*, the procedure led the core to the origin along the curve *I*. The lowest point of the curve *VQAG* lies, of course, nearly over the point *Z*. The shaded diagram in the upper part of the figure shows a similar curve obtained at another time and drawn strictly to scale. If after many reversals of a comparatively small current the core which started at *L* reached the point *F*, and if the current was then slowly increased, the core made the journey indicated by the line *FL*. The shaded diagram in the lower part of the figure is a reduction of a curve obtained with a large induction coil the core of which is a compact round bundle of fine wire 7.5 cms. in diameter and about 85 cms. long. The curves *oec*, *cak*, *cek*, in this diagram correspond to *OIQV*, *VPG*, *VQAG* in the larger figure. The retentiveness of a core of these dimensions is, of course, very small.

Even if much time has been spent in demagnetizing a large closed core by sending through the exciting coil currents alternately in one direction and in the other, of intensities gradually decreasing to a very small final value, it frequently happens that after a much larger current has been put for, say, twenty times through the coil alternately in one direction and the other, the hysteresis cycle does not "close," for the change of flux caused by applying the given current

in one direction is not equal to the flux change caused by applying the same current in the other. This fact often makes the accurate determination of a hysteresis diagram for such a core a long and trying piece of work. Some toroidal cores I have never succeeded in demagnetizing completely. The demagnetizing apparatus which I have usually employed in the course of the work here described consists first of a storage battery of forty large cells, a set of rheostats made up of metallic and liquid resistances intended for heavy currents, and a commutator run from the main shaft of the laboratory machine shop, and so arranged as to reverse the direction of the current from the cells every ten seconds. Starting with no resistance in the rheostats, resistance was gradually introduced into the circuit until the current had become very small. After this procedure, the secondary circuit of a specially constructed transformer was attached to the exciting coil of the magnet, and from an initial voltage of about 660, at 60 cycles per second, the electromotive force was gradually decreased until the current became too small to measure. In some cases it seemed better to omit the second part of the process.

The Establishment of a Steady Current in the Coil of an Electromagnet

If the circuit of the exciting coil of an electromagnet contains a battery of storage cells of constant voltage E, and if this circuit be suddenly closed, the strength of the current will rise more or less gradually from its initial zero value to E/r amperes, where r is the whole resistance of the circuit in ohms. In the case of a given magnet, with a given electromotive force in the coil circuit, the manner of growth of the current depends very largely, as we have seen, upon the magnetic state of the core when the circuit was closed. The three curves of Figure 19, which are carefully made reproductions of the photographed records of an oscillograph, show the march of the current from a battery of 20 storage cells in the circuit of a coil of 2788 turns belonging to the magnet Q under three different sets of conditions. If after the core had been demagnetized as thoroughly as possible, by the method already described, the circuit was suddenly closed, the current followed the middle curve of the three. If the current was allowed practically to attain its maximum value, and if then

a commutator in the circuit was reversed and, at intervals of a few seconds, reversed again and again, and if finally the circuit was broken, it was possible by closing the commutator again in the proper direction, to make the new current follow either the upper or the lower curve of the diagram. If this current coincided in direction with the last current through the coil, the current was "direct," and its rise was represented by the upper curve. If the new current had a direc-

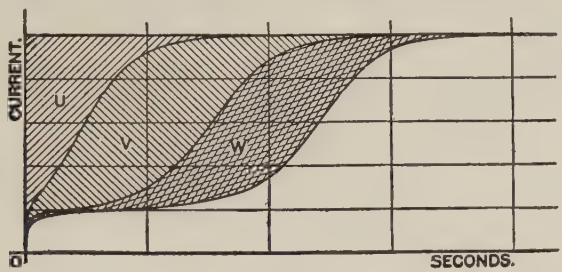


FIGURE 19. — Currents from a battery of 20 storage cells in the circuit of a coil of 2788 turns belonging to the magnet Q. Before the middle curve was taken, the core was carefully demagnetized. The upper and lower curves represent direct and reverse currents, respectively. The areas V and W are equal.

tion opposite to that of the last current through the coil, the current was "reverse," and followed the lower curve. The areas V and W are practically equal.

It is evident that, other things being equal, the rapidity of rise of the current in a circuit which contains a coil wound around the core of an electromagnet will depend very much upon the number of turns in the coil. Figure 20[1] shows reverse curves from the magnet R. The actual strengths of the currents were 6, 3, and 1.5 amperes respectively, and the numbers of turns in the exciting coils were 85, 170, and 340. The electromotive force was the same in all three cases. The horizontal units are tenths of seconds.

Although the typical current curve for the coil of an electromagnet wound in many turns about the core has two points of inflexion if the core is laminated, both of these disappear if the change of the magnetic flux through the circuit due to the current is small enough, and occasionally one finds an oscillogram which seems to have only one

[1] Identical with Fig. 12 on p. 74.

point of inflexion. Some of the direct curves shown in Figures 5, 23, and 28 are everywhere convex upward. Among the nearly three thousand photographed oscillograph records taken for use in this paper no one is concave upward at the very start, but a curve of this kind, with one point of inflexion, has been shown by Dr. Thornton, and I have many curves which become concave upward very near the origin. In current curves belonging to the coil of an electromagnet which has a large closed, solid core, there are often two points of inflexion, but many of even the reverse curves are everywhere convex upward. Figure 21 [1] shows curves taken for the coil of the large magnet P in the circuit of which was a storage battery of voltage 84. When each current started, the core was nearly neutral.

When the coil of a transformer, the core of which is built up of such thin plates of soft iron as are used in the best practice, is subjected to an alternating electromotive force of extremely high frequency, the disturbing effect of eddy currents in the iron are, of course, very apparent, but the manner of growth of a current under a constant electromotive force is usually not very greatly affected by such currents.

The fact that the susceptibility of the iron is by no means constant, materially alters the shape of a current curve when iron is introduced into a circuit; nevertheless, it is instructive to compare the manner of growth of a current in the coil of an electromagnet which has such a core, with that of a current in a circuit of fixed inductance, without attempting at the outset to account mathematically for the differences, though it will be easy to do so later on.

In the case of a simple circuit, without iron, of resistance r ohms and *constant* inductance, L henries, which contains a constant electromotive force of E volts, the rise of the current I when the circuit is suddenly closed follows the law

$$I = \frac{E}{r}(1 - e^{-\frac{rt}{L}}), \tag{16}$$

and attains the fractional part k of its final value (E/r) in the time

$$t = -\frac{L}{r} \cdot \log_e(1 - k), \tag{17}$$

which is independent of the ultimate current strength and involves only the time constant (L/r) of the circuit. If the circuit is made up

[1] Identical with Fig. 13 on p. 75.

partly of non-inductively wound resistance wire, and partly of helices, r may be kept constant, while L is changed, by changing the relative proportions of the two parts; or r may be altered while L is constant, by increasing or decreasing the non-inductive portion of the circuit.

If E/r and L are given, different values of E may be used by giving properly corresponding values to the non-inductive resistance, and if the "building-up time" of the current under given initial conditions in the core be defined as the number of seconds required for the current to attain any arbitrarily chosen fractional part of its final value,

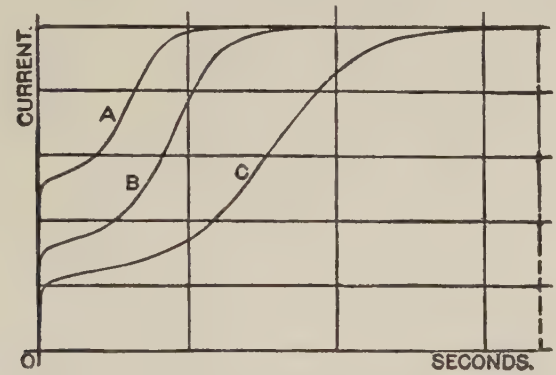


FIGURE 22. — Curves which show the manner of growth of currents in a coil of 1394 turns belonging to the magnet Q, to a given final value, when the applied voltages were 82, 41, and 20.5, nearly. In each case the core was neutral at the outset.

this time will be inversely proportional to E. In the case of a circuit which has one or more iron cores the phenomenon is much less simple, and if the cores be of solid metal, the effects of eddy currents may complicate the problem seriously; but although under these circumstances the law of proportionality no longer holds, it is almost universally true that the establishment of a current of given final intensity in the coil of a given electromagnet can be accelerated by increasing very much the applied electromotive force and then introducing a sufficient amount of non-inductive resistance to make E/r the same as before.

Figure 22 shows current curves for the magnet Q under a fixed final excitation of 2650 ampere-turns. In curves A, B, C, the currents were

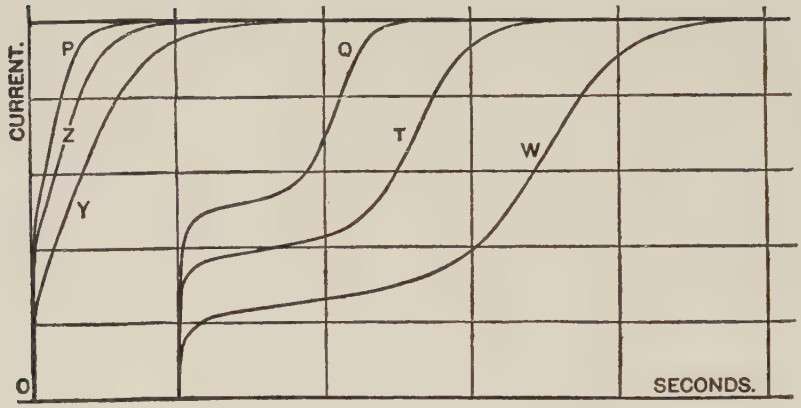


FIGURE 23. — Direct and reverse current curves for the magnet Q with a given final excitation of 2650 ampere turns, under applied voltages of 82, 41, and 20.5, nearly.

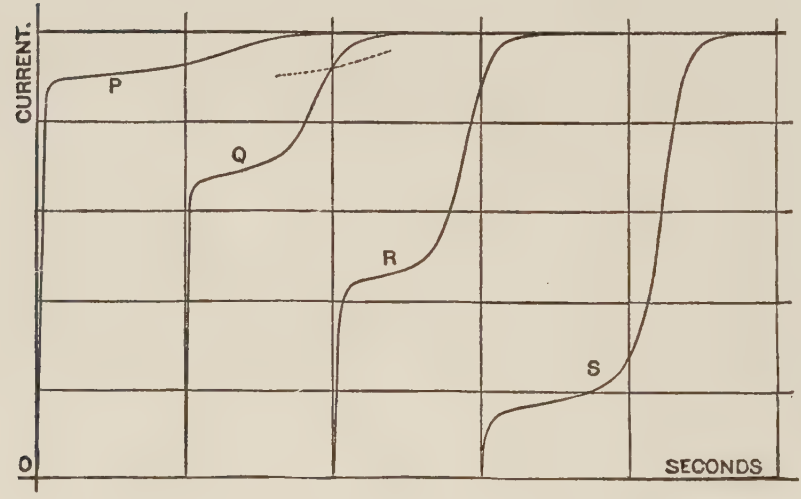


FIGURE 24. — The manner of establishment of a current of final strength 2.60 amperes, in the coil circuit of the magnet Q, under a voltage of 82, when the number of active turns was 407, 823, 1394, or 2788.

caused by 40 cells, 20 cells, and 10 cells, respectively, and these currents were made equal by adding to the circuit in each case a suitable non-inductive resistance. Before each of these curves was taken, the core of the magnet was carefully demagnetized by the elaborate process described above. After the magnet Q had been put a good many times through a cycle with a given maximum excitation of 2650 ampere turns, under one of the voltages just named, direct and reverse

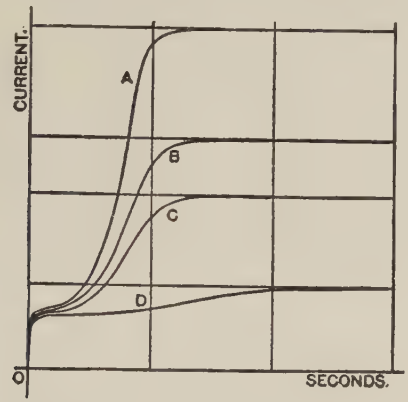


FIGURE 25. — Currents in the coil of the electromagnet Q for four different values of r when E and the number of magnetizing turns were fixed. At the starting of each current the core was magnetically neutral.

curves were taken with the help of the oscillograph. Careful reproductions of these curves are given in Figure 23: to avoid confusion the reverse curves are drawn from a separate time origin.

If in a circuit which contains no iron, E and r be kept constant, while L is changed, the building-up time as defined by equation (17) will be proportional to L. Of course no such simple relation holds when the circuit includes the magnet Q; Figure 24 shows current curves for the same final value of 2.60 amperes, under an applied electromotive force of about 82 volts, for exciting coils of 407 turns, 823 turns, 1394 turns, and 2788 turns. For convenience, the curves are drawn from different time origins. The dotted line which crosses curve Q calls attention to the fact that if curves P and Q were drawn from the same origin, the former would cross the latter.

If in a circuit without iron E and L were kept constant while r was varied, the building-up time (L/r) would be inversely proportional to the resistance of the circuit, or, since the electromotive force is fixed, directly proportional to the current strength. There is no approximation to this in a circuit which contains iron. The current curves shown in Figure 25 were obtained from the electromagnet Q when 2788 turns

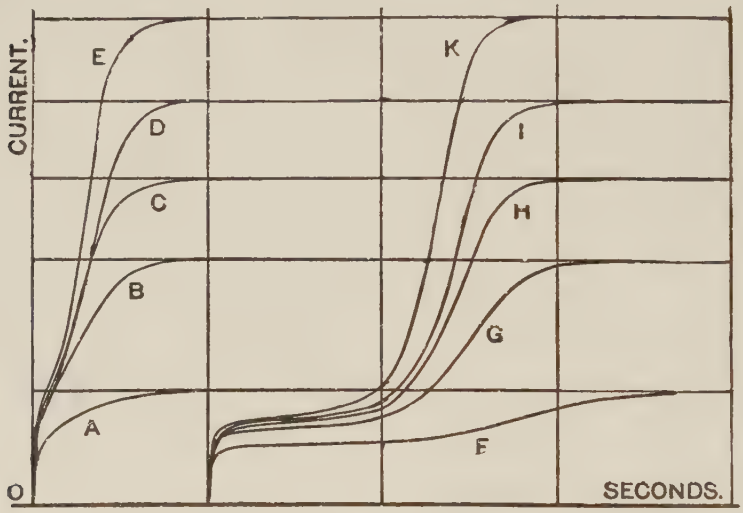


FIGURE 26. — Direct and reverse current curves in the coil of the electromagnet Q for five different values of r when E and the number of active turns were kept fixed.

were used in the exciting coil and a battery of 40 storage cells with a voltage of about 82 furnished the electromotive force. Curve C evidently corresponds to a case where the total resistance in the circuit is about twice as great as in the case represented by A, but for every value of k the building-up time is greater for C than for A, though the difference becomes very small at the end. A comparison between A and D shows the same fact. Before each of the curves A, B, C, D, was taken the core of the magnet was carefully demagnetized. Figure 26 exhibits current curves taken for different values of r with the same coil of the magnet Q and with the same electromotive force as the curves just mentioned. In each of the cases shown in Figure 26 the core was put several times through a cycle before the direct and reverse

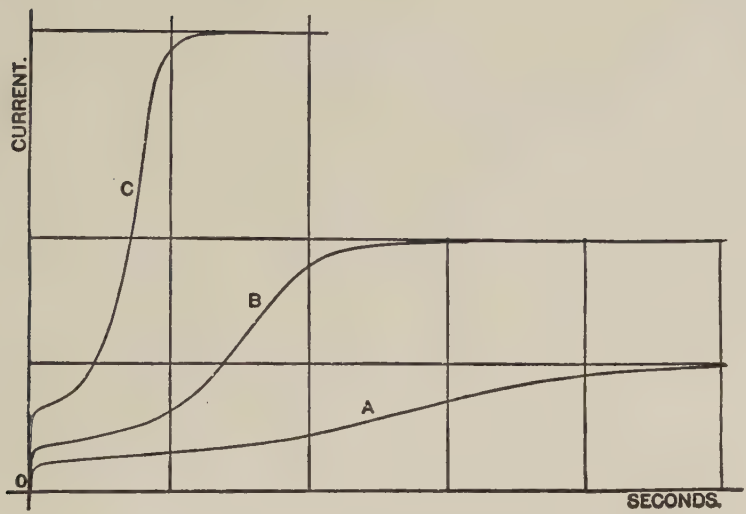


FIGURE 27. — Currents in the coil of 2788 turns belonging to the magnet Q for three different values of the applied voltage with the same value of r. At the starting of each current the core was magnetically neutral.

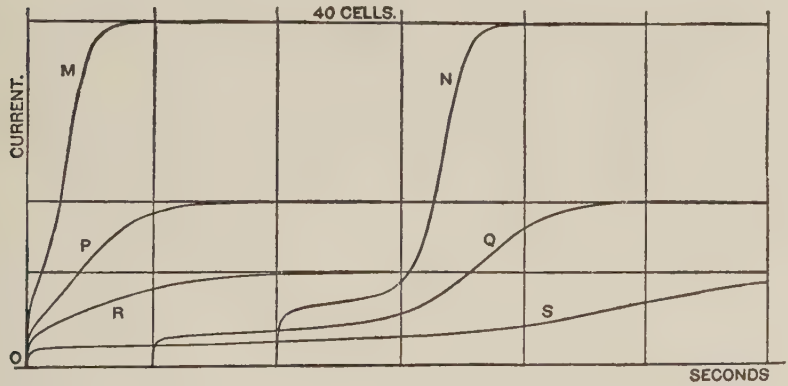


FIGURE 28. — Direct and reverse currents in a coil of 2788 turns belonging to the magnet Q for three different values of the applied voltage, but the same value of r.

oscillograms were taken. The records are reproduced as accurately as possible; B, C, and D run together in a complicated manner, and the same tendency is shown in the reverse curves G, H, I, but in general the longer building-up times belong to the lower currents.

If in an inductive circuit without iron r and L are fixed, the building-up time will be independent of the value of E, but this is not the fact if the circuit contains an electromagnet. Figures 27 and 28 show current curves obtained from the coil of 2788 turns belonging to the magnet Q. In all the curves of each diagram the value of r was the same, but the voltage of the battery in the coil circuit had three different values the largest of which (belonging to the curves C, M, N) was about 82: in this case the current was almost exactly 2.50 amperes.

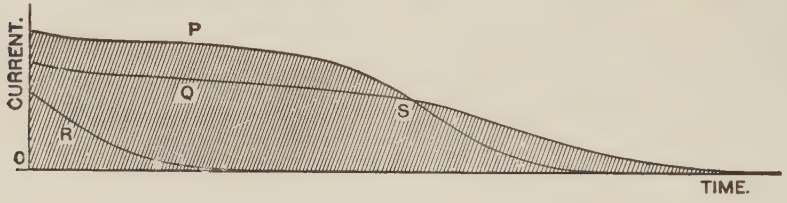


FIGURE 29

Before each of the curves A, B, C was taken the core was thoroughly demagnetized: R, P, M are direct curves, but S, Q, N are reverse curves. It is evident that the building-up times are not even approximately independent of E.

Figure 29 shows the records of an oscillograph in a secondary circuit in which were a few turns of wire wound around the core of the magnet Q. The primary circuit contained, besides the storage battery, a rheostat and an exciting coil of 1394 turns. When the primary circuit was suddenly closed with such a resistance in the rheostat that the final strength of the current was 1.1 amperes, the induced current had the value indicated by the curve Q; when the rheostat resistance was suddenly removed so as to bring the final strength of the current up to 2.3 amperes, the induced current curve was R. The sum of the areas under the curves Q and R was 74.3 square centimeters. The curve P shows the current record in the secondary circuit when the primary circuit was suddenly closed with no resistance in the rheostat: the area under this oscillogram was 74.6 square centimeters. All

the currents were reverse currents. Most of the area determinations of this paper were made with a Coradi "Grand planimètre roulant et à sphère."

Figure 30 shows a careful reproduction of the record of an oscillograph in the primary circuit of the arrangement just described. These curves were taken on the same day as those of the last figure. In this case the flux change due to the current which gave the curve T was

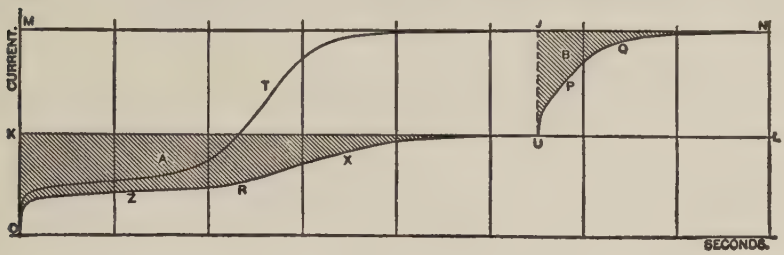


FIGURE 30

to the sum of the flux changes caused by the partial currents as 1130 to 1126. These numbers do not show any real difference between the corresponding physical quantities, but point to difficulties of measurement.

THE EFFECT OF THE MAGNETIC CHARACTERISTICS OF THE CORE UPON THE MANNER OF GROWTH OF A CURRENT IN THE COIL OF A LARGE ELECTROMAGNET

If under the application of a constant electromotive force to the coil circuit of an electromagnet a current grows gradually in the coil to its full value, the magnetic flux in the core at any moment depends, as we have seen, not only upon the instantaneous strength of the current, but also upon the magnetic state of the core at the beginning. Moreover, if the core is solid, it is clear that the magnetizing field to which the interior of the iron mass is exposed may be quite different at any instant from what it would be if eddy currents were non-existent. If, however, the core is built up of such thin sheets of iron as are used in good transformers, a fair approximation to the form which the current curve will have under any given circumstances can be made if one has an accurate statical hysteresis diagram of the core for the

range required, and if the core is made of very fine varnished wire, as in the case of loading coils for long telephone circuits, a hysteresis diagram obtained either from a long "step-by-step series" of measurements or from one or more oscillograms, enables one to predict with accuracy what the form of a current curve will be for any practical case. These last statements are based on experiments such as those recorded below.

As a result of a long series of measurements, it appears that when the core of the magnet Q has been well demagnetized and a series of steady currents each a little stronger than the preceding one are established in the exciting coil, the magnetic flux through the core in thousands of maxwells follows fairly accurately the course indicated in the following table:

TABLE I

Ampere Turns	Magnetic Flux	Ampere Turns	Magnetic Flux
100	35	1100	1208
200	146	1200	1238
300	386	1300	1262
400	622	1400	1285
500	787	1500	1309
600	929	1600	1331
700	1013	1700	1352
800	1086	1800	1369
900	1137	1900	1390
1000	1176	2000	1409

Figure 31 reproduces the table graphically in the full curve: the vertical unit is a thousand maxwells, and the horizontal unit is 139.4 ampere-turns, to suit the case when the particular exciting coil used has 1394 turns. The ordinates of the dotted curve represent twice the

TABLE II

Current in Amperes	Log [(13.94)λ]	Current in Amperes	Log [(13.94)λ]
0.00	0.445	0.55	1.135
0.05	0.860	0.60	1.025
0.10	1.248	0.65	0.943
0.15	1.602	0.70	0.860
0.20	1.715	0.75	0.797
0.25	1.672	0.80	0.746
0.30	1.594	0.90	0.700
0.35	1.496	1.00	0.635
0.40	1.399	1.10	0.621
0.45	1.312	1.25	0.606
0.50	1.209	1.30	0.591

corresponding values of the slope (λ) of the other. A template of the curve B was made as accurately as possible from a large piece of sheet zinc, this was fastened down on a table over a number of sheets of co-ordinate paper, and the value of λ was determined by measuring on the paper the position of a straight edge which touched the template at any desired point.

If after the core of Q had been demagnetized, a steady electromotive force of E volts were applied to the exciting circuit of resistance r

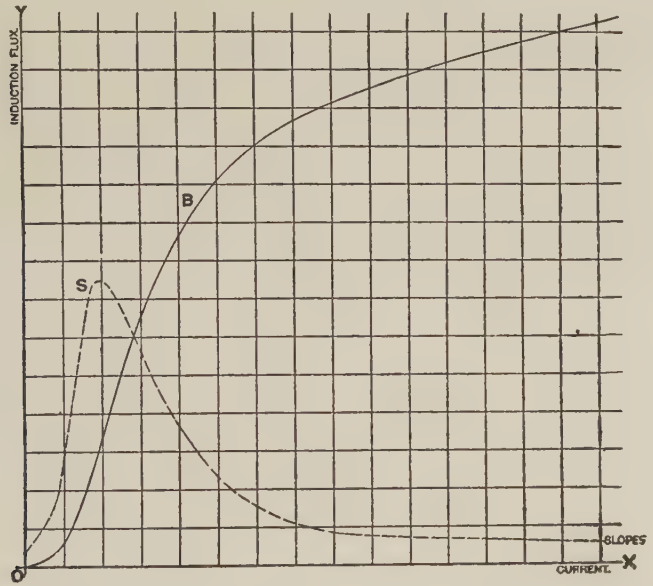


FIGURE 31. — Magnetization curve for the core of the magnet Q which at the outset is in a neutral state. The ordinates of the dotted curve represent twice the slopes of the other curve.

ohms, containing the coil of 1394 turns, and if eddy currents were non-existent so that the core followed the statical magnetizing curve, the march of the current (in amperes) would be given by the equation

$$E - ri = 13.94\lambda \cdot \frac{di}{dt}, \tag{18}$$

whence

$$t = \int_0^i \frac{13.94\lambda}{E - ri}\, di. \tag{19}$$

If from an actual current curve obtained from Q for a given journey of the core we were to determine the corresponding magnetizing curve for the metal (flux versus coil current), we should find that the values of the flux, for small values of the current, at least, would fall short of the flux values which the same currents would cause if they were to act for some time because the magnetizing field is less than that due to the coil current by that due to the eddy currents. If, therefore, from the numbers of Tables I and II we were to determine the form of a

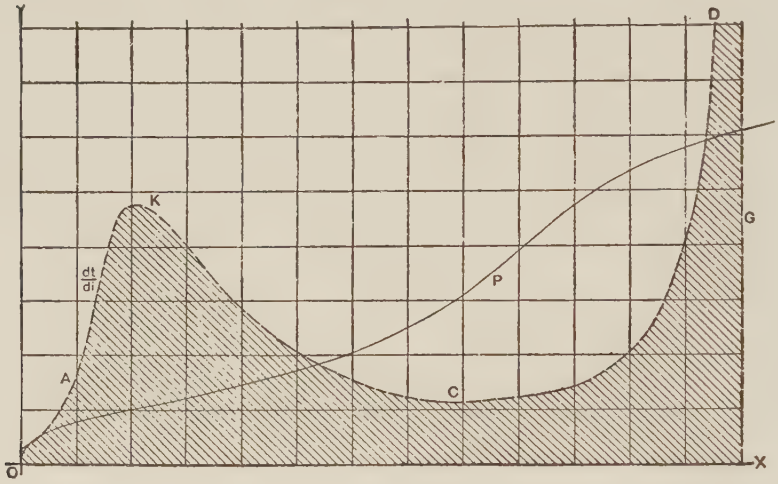


FIGURE 32.—The ordinates of the boundary of the shaded area represent $2\,(dt/di)$ for $E = 26$, $r = 20$. P shows the theoretical form of the corresponding current curve.

current curve for Q, corresponding to any journey of the core, this would fall somewhat below the actual curve at the beginning. The core of Q has, however, a typical magnetizing diagram, and the theoretical curves are instructive as showing what the actual curves would be if the same core were more finely divided. The effect of eddy currents can be seen in the curves for this magnet given above.

The boundary of the shaded area in Figure 32 shows twice the value of the integrand

$$w = \frac{13.94\lambda}{E - ri} \tag{20}$$

for the case $E = 26$, $r = 20$: the horizontal unit is one tenth of an ampere. The vertical line corresponding to $i = 1.3$ is evidently an

asymptote. The area under the curve from the beginning to the ordinate representing any given value of the current shows, in twentieths of a second, the time required, under the given conditions, after the circuit is closed for the current to attain this value. It is easy to determine a series of such areas with the help of a good planimeter, and the full curve of Figure 32 actually represents the growth of the current in the case mentioned according to my measurements of the large diagram of which Fig. 32 is a very much reduced copy: for this curve

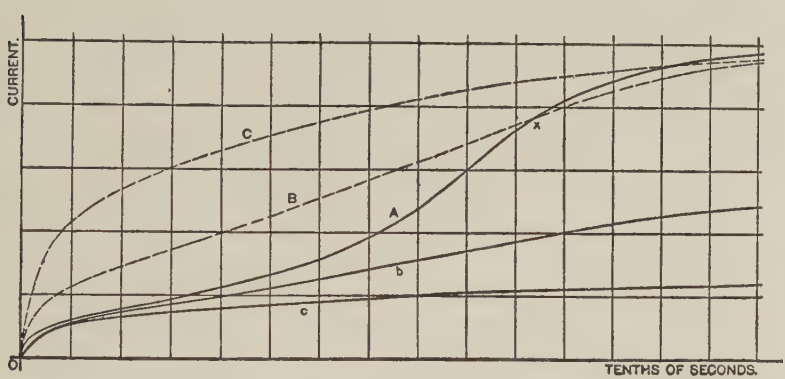


FIGURE 33. — Forms of current curves for Q deduced from theoretical considerations. The coil has 1394 turns and contains a storage battery of voltage 26. C is everywhere convex upward: A and B have two points of inflexion.

the horizontal unit is one tenth of a second and the vertical unit is one fifth of an ampere. This curve has the general form of most of the current curves which one obtains with a transformer the core of which is at the outset neutral, but it is evident that in any case where the final value of the current is small enough the asymptote will be moved so far to the left that the integrand curve will rise continually from the beginning, without the maximum and minimum values, and the current curve will have the everywhere convex shape that we find in practice when we cause the current to grow by short steps in the manner indicated by the curve U in Figure 4.

Figure 33 shows building-up current curves (A, b, c) for $E = 26$, and $r = 20$, 40, and 60, respectively. The dotted curves B and C are copies of b and c with ordinates so magnified that the curves have the same asymptote as A. According to this diagram the current attains

75 per cent of its own final value more quickly when r is 40 than when r is 20, but B crosses A at the point x and the current seems to reach practically its full strength sooner in the latter case. The curve C first crosses the curve A and then B. It would be easy to show from a series of oscillograph records for similar cases that the characteristics of the theoretical curves correspond in general to fact.

If with the core of the magnet Q initially neutral a steady current of given strength be established in the coil of 1394 turns, by use of a storage battery of voltage E, the integrand will be for every value of the current inversely proportional to E (since E/r is given), and the

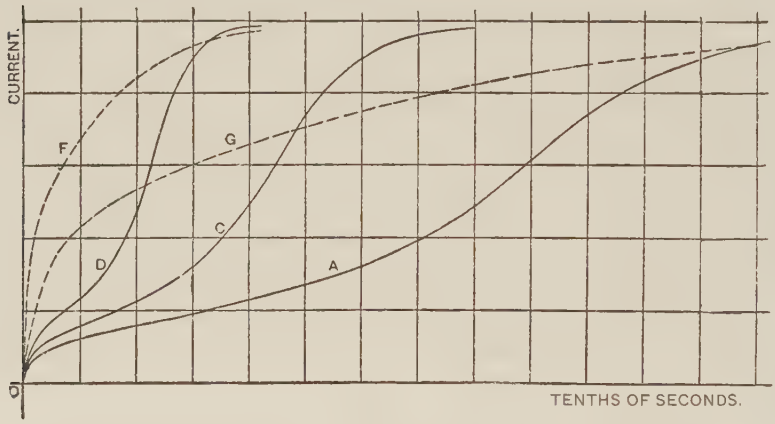


FIGURE 34. — Theoretical forms of current curves in a coil of 1394 turns belonging to the magnet Q. In practice these would be somewhat modified by eddy currents.

building-up time will be inversely proportional to the applied electromotive force, as it would be if the inductance were fixed. For a given exciting coil, the general shape of the curve for a given current is independent of the applied voltage. Curves A, C, and D of Figure 34 are the current curves computed for $E = 26, 52, 104$, and $r = 20, 40$, and 80: the maximum value of the current is the same in every case. G and F are the current curves computed for $E = 26$, $r = 80$, and for $E = 104$, $r = 320$.

As has been explained already, it is difficult to obtain an accurate hysteresis diagram for a very large core by the ordinary ballistic methods with such galvanometers as are usually to be found in the

testing room, but it is fairly easy to attach extra weights to the suspended system (Figure 35) of a good d'Arsonval or Thomson Mirror galvanometer which shall so increase the moment of inertia that the time of swing shall be lengthened to five or ten or twenty minutes. With an instrument thus modified it is usually possible, by changing the intensity of the current in the exciting coil by small steps, to deal satisfactorily with very large masses of iron. It is of course desirable to use a rather high electromotive force in the exciting coil in order to make the building-up time short, and to reduce the current to the desired strength by introducing extra non-inductively wound resistance into the external circuit. In order to test this matter thoroughly, I measured with great care, by aid of a modified Rubens-du Bois "Panzer Galvanometer," the flux changes in the core of the magnet Q (the area of the cross-section of which is more than 150 square centimeters), corresponding to a hysteresis cycle for an excitation of 1812 ampere turns. I then determined the same total flux change by means of planimeter measurements of the areas under a long series of oscillograph records; all the testing instruments were different in the two cases, and no comparison was possible until the final results were obtained and were found to differ from each other by only one part in about fourteen hundred. The labor of reducing the oscillograms was very great, and this extremely close agreement must be considered accidental, since it is not easy to make a large mass of iron go over exactly the same magnetic journey twice.

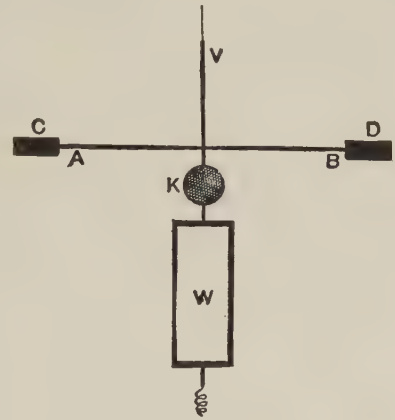


FIGURE 35. — The horizontal rod AB is threaded and the brass masses C, D can be screwed on the rod as far as is necessary. The system must be accurately balanced.

Hysteresis diagrams for the magnet Q and corresponding to maximum excitations of 1812, 5370, and 10,880 ampere turns are given in Figure 36. Some results of measurements of the flux changes in the core for the first of these cycles are given in Table III.

TABLE III

Ampere Turns	Flux in Thousands of Maxwells	Ampere Turns	Flux in Thousands of Maxwells
1812	1371	−131	772
1394	1351	−148	734
1255	1340	−181	552
1031	1316	−234	332
809	1285	−294	22
474	1211	−392	−465
392	1186	−474	−661
294	1148	−809	−1010
234	1121	−1031	−1128
181	1099	−1255	−1214
148	1070	−1394	−1265
131	1060	−1812	−1371
000	953		

After a curve had been drawn on a very large scale to represent the numbers of Table III, a zinc template was made from it, by aid of which and a long "straight-edge" the slopes of the curve could be determined with some accuracy. The next diagram (Figure 37) shows the slope as a function of the strength of the current.

When the slope for any point of the curve is multiplied by (13.94) / $(E - ri)$, where E and r are given, the result is the value of dt/di for

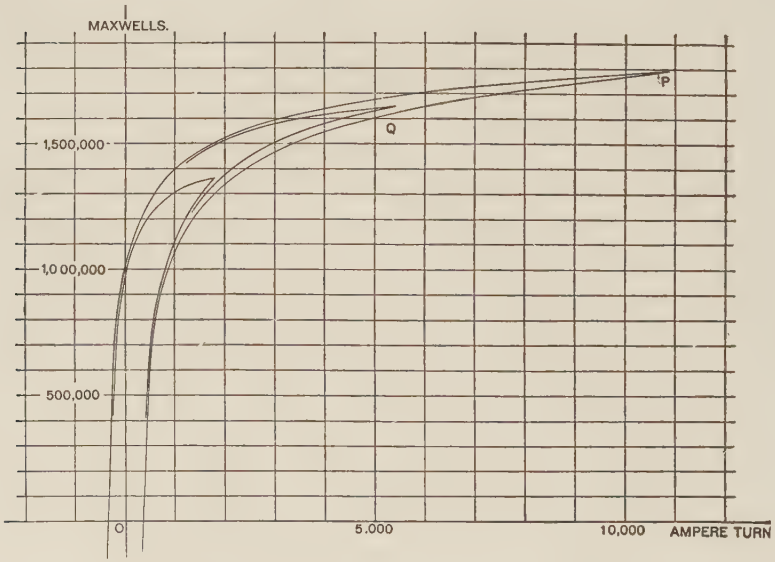


FIGURE 36. — Hysteresis diagrams for the core of the magnet Q.

the reverse current curve when the applied voltage is E and the resistance r, for the given value of i. Figure 38 exhibits dt/di for $E = 19.5$, and $r = 15$.

The actual curve was drawn on a large scale, and the area X from $x = 0$ to $x = i$, for a number of different values of i were measured by a planimeter in terms of the unit square of the figure; this area expressed in tenths of seconds the time required for the reverse current to attain the strength i. A few values of X are shown in the next table.

TABLE IV

i	$X/10$	i	$X/10$
0.05	0.057	0.50	1.750
0.10	0.155	0.60	1.875
0.15	0.494	0.70	1.985
0.20	0.878	0.80	2.088
0.25	1.141	0.90	2.188
0.30	1.325	1.00	2.294
0.35	1.471	1.10	2.412
0.40	1.579	1.20	2.632

Every form of current curve which I have met in practice can be closely imitated by a theoretical curve; but all these curves have

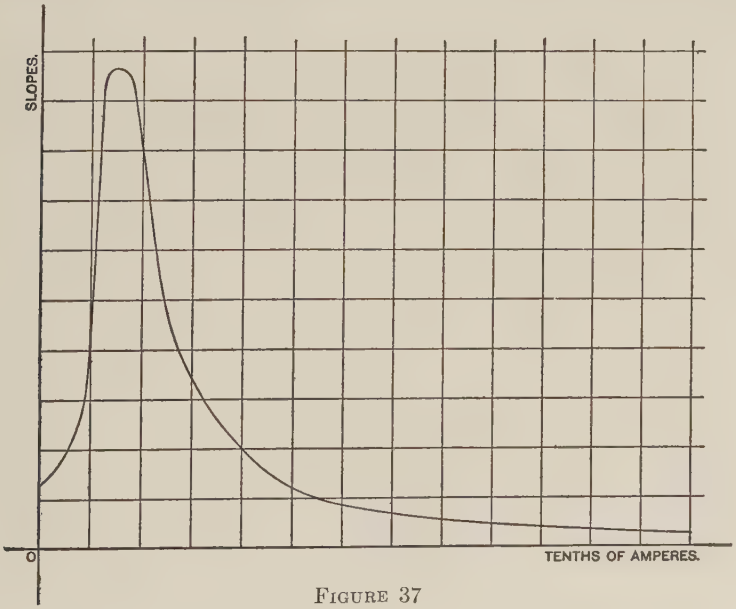


FIGURE 37

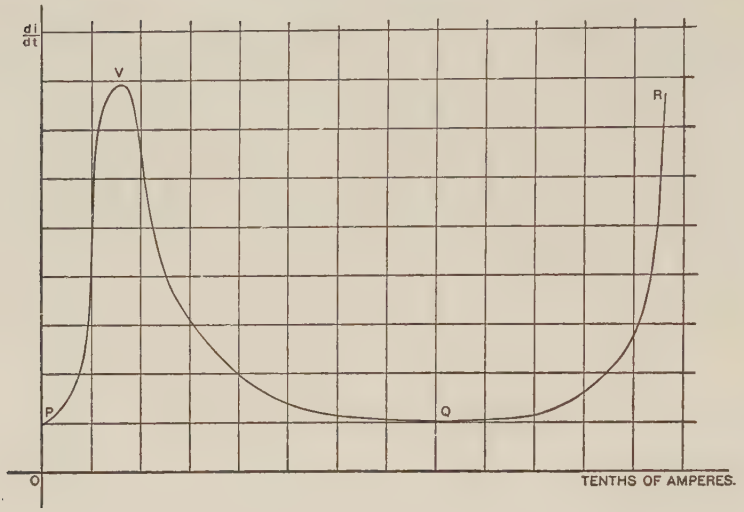


FIGURE 38. — The value of dt/di for a reverse current in a coil of the magnet Q when $E = 19.5$ and $r = 15$.

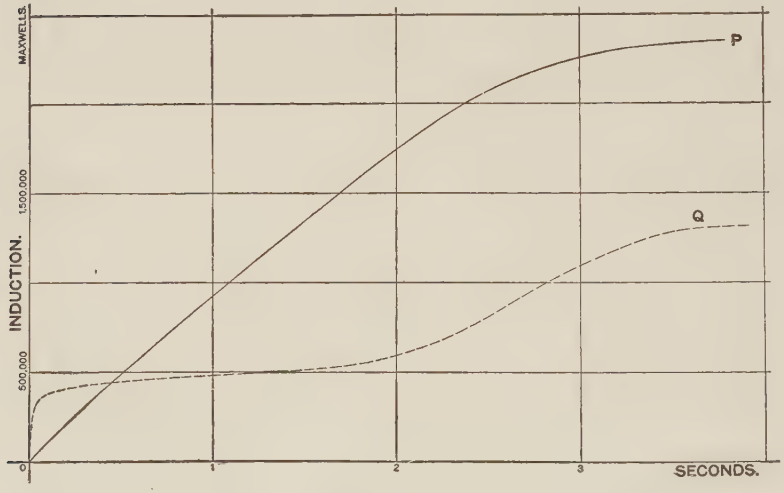


FIGURE 39. — The full curve shows the rate of increase of the flux of magnetic induction through the core of the magnet Q while a reverse current of 1.3 amperes is being established in the exciting coil of 1394 turns. The current curve is shown on an arbitrary scale by the dotted line.

at the outset a direction differing widely from the horizontal. Dr. Thornton, however, shows a beautiful curve which at the beginning is convex downward and has at the start a direction not very different from that of the axis of abscissas.

Before one uses an oscillograph for purposes of accurate measurement, one must make sure that the instrument has been properly set up. When the drum which carries the sensitive film or paper is at rest, a current sent through the conductor should give a perfectly straight record accurately perpendicular to the base line, and the length of this record should be proportional to the strength of the

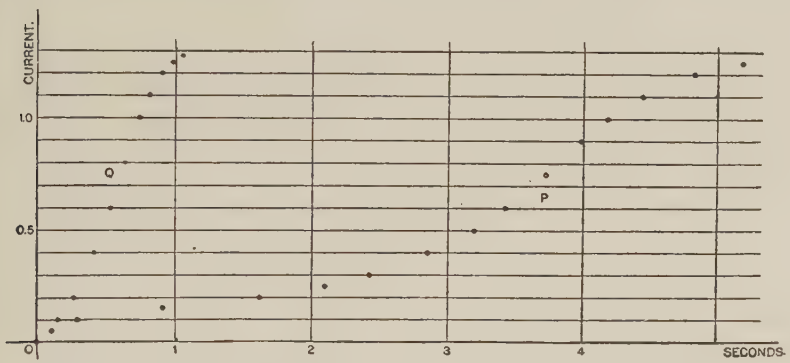


FIGURE 40. — Theoretical forms of direct and reverse current curves for a coil of 1394 turns belonging to the magnet Q when the resistance of the circuit is 8 ohms and the applied voltage is 10.4.

current. It sometimes happens that an oscillograph which records accurately the march of a moderate current lags in its indications a very little behind the strength of a comparatively feeble current owing to the viscosity of the oil used for damping, which only then becomes troublesome. I have myself had sad experience in drawing from the records of an instrument of this sort, which I thought I had carefully calibrated, elaborate inferences which were contrary to fact. If, however, one has at hand, first, a well-constructed and mounted ballistic galvanometer with a period of from eight to ten minutes, and means of damping the swings of the suspended system (electromagnetically or otherwise) without touching it, and secondly, some kind of chronograph designed to close and after a given interval to open again any circuit to which it may be attached, it is easy to test almost any sup-

posed fact about the growth of the flux through the core of an electromagnet.

The toroids I used had cores made of extremely fine, varnished iron wire, costing about four dollars per kilogram. For some of these I determined by ballistic methods, as carefully as I well could, the hysteresis diagrams for several excitations, and then compared with these other diagrams obtained from the oscillograph records of current curves for the same magnetic journeys of the cores, but I could not

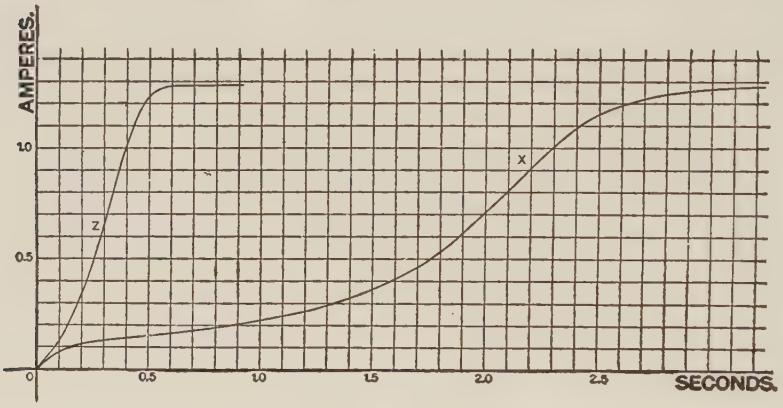


FIGURE 41. — Theoretical forms of direct and reverse current curves for a coil of 1394 turns belonging to the magnet Q when the resistance of the circuit is 15 ohms and the applied voltage is 19.5.

detect any differences which did not lie far within the small uncertainty which the viscosity of the oil in the oscillograph may be supposed to cause. It does not seem worth while to print a long series of numbers to illustrate this kind of comparison though the labor was great.

If, then, the core of an electromagnet is made of iron wire not more than one tenth of a millimeter in diameter and carefully varnished, it seems to be true within the limits of accuracy of my measurements and *for the comparatively moderate excitations used*, that if the core is in a given magnetic state at the start, the change of the flux of magnetic induction caused by a current which grows from zero without decreasing to a given final intensity, is quite independent of the manner of growth of this current. It may grow continuously or by

steps, and if eddy currents are not appreciable, the condition of the core at the end is the same. According to this, one would get exactly the same hysteresis diagram from an accurately drawn current curve of the form V (Figure 4) corresponding to any change of current in the exciting coil as from the corresponding U diagram or from any slow

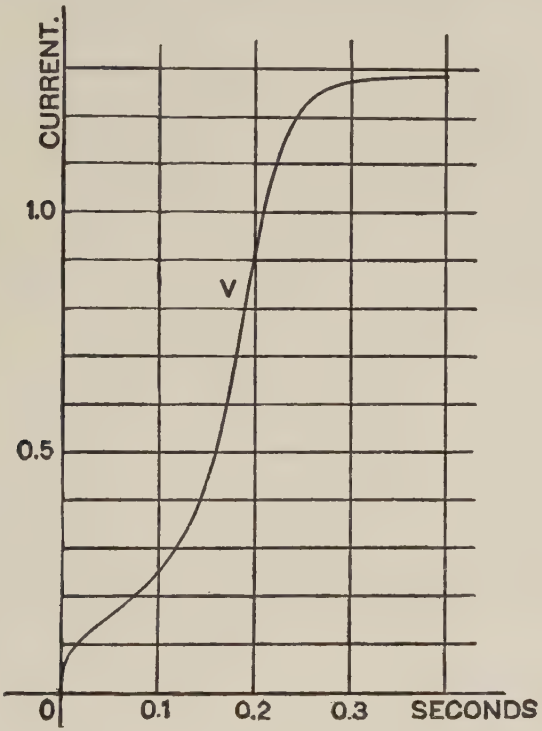


FIGURE 42. — Theoretical form of reverse current curve for a coil of 1394 turns belonging to the magnet Q, under an electromotive force of 208 volts. The resistance of the circuit is 160 ohms.

step-by-step ballistic method. Nothing of the nature of time lag, if it exists at all, affects the growth of the induction in the iron appreciably. Even in the case of an ordinary transformer, where the effects of eddy currents are very noticeable at the early portions of most current curves, the whole change of flux due to a given current in the coil is the same apparently whether the current grows steadily or by steps; in this case an accurate diagram of the U form and a step-by-

step ballistic method with a proper galvanometer may be expected to yield identical results within the limits of the measurements. This statement seems to be justified by such comparisons of the two as that recorded on page 141, which required many days in the making. From a current curve we may expect to get a hysteresis diagram good enough for any commercial purpose, but differing slightly at the beginning from the statical diagram found ballistically. Of course, it would not be easy to get any very accurate information, as some of the curves given in this paper show clearly, from a current curve taken in the exciting coil of a magnet which has a large solid core.

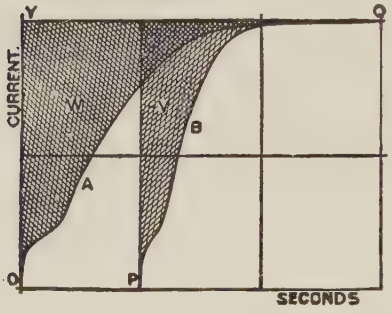


FIGURE 43

It will be evident from what precedes that it is possible to predict accurately the building-up curve of a current in the coil of an electromagnet with fine wire core, from a corresponding hysteresis diagram obtained by aid of a ballistic galvanometer of long period, and one of the old methods of procedure.

Figure 43 shows two reverse current curves for a toroidal magnet of about one third of a henry inductance belonging to the American Telephone and Telegraph Company. The final strength of the current was the same (1.42 amperes) in both cases, but the applied electromotive force was 10.9 for the left-hand curve and 21.5 for the other. The disturbing effects of eddy currents were here (as will be shown in the sequel) wholly inappreciable. We should be justified in expecting that each of these current curves would yield by aid of a good planimeter a hysteresis diagram substantially the same as any ballistic step-by-step method would furnish for the same magnetic journey of the core.

The Influence of Eddy Currents upon the Apparent Magnetic Behavior of the Core of a Large Electromagnet in the Coil of which a Current is growing

If after the solid core of a large electromagnet had been demagnetized we were to establish a steady current in the exciting coil by applying to its circuit a constant electromotive force, eddy currents would, of course, be set up in the core, and at any instant during the growth of the current in the coil the iron at the centre of the core would be subjected to a magnetic field weaker than the field belonging to a steady current of intensity equal to the instantaneous strength of the coil current. If, therefore, we were to attempt to determine the magnetic properties of the core from the record of an oscillograph in the coil circuit, we should find that the induction through the core corresponding to a given instantaneous current intensity in the coil was less than the flux belonging to a steady current of the same intensity as determined from a statical hysteresis diagram. The same phenomenon appears when an electromagnet with finely laminated core has a secondary coil. The closing on itself of a secondary coil wound on the core of an electromagnet when a current is being established in the primary will, therefore, expedite at first the rise of this current, but the area over the current curves ought to be the same in the two cases, and we must expect, therefore, the building-up time to be somewhat longer when the secondary coil is closed than when its circuit is broken.

It is to be expected, of course, that the curves which show the march of the current in the primary circuit will be noticeably different in form when the secondary circuit is closed and when it is open; for this is often the fact in the case of two neighboring circuits which have *fixed* self and mutual inductances (L_1, L_2, M) if one of them containing an electromotive force E be suddenly closed at the time $t = 0$, while the other, which contains no electromotive force, is closed. Here

$$\begin{aligned} L_1 \cdot \frac{dI_1}{dt} + M \cdot \frac{dI_2}{dt} + r_1 \cdot I_1 &= E_1, \\ M \cdot \frac{dI_1}{dt} + L_2 \cdot \frac{dI_2}{dt} + r_2 \cdot I_2 &= 0, \end{aligned} \tag{21}$$

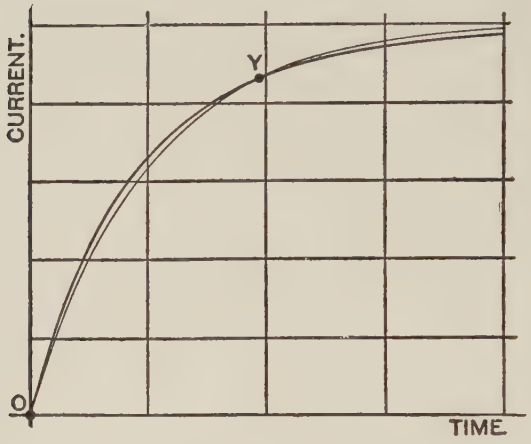


FIGURE 44. — Currents in the primary circuit of an induction coil with air core, when the secondary circuit is closed (full curve) and when the secondary is open.

where r_1, r_2 are the resistances of the circuits and I_1, I_2 the currents in them.

If $$\lambda = -\frac{(Q-R)}{2S}, \quad \text{and} \quad \mu = -\frac{(Q+R)}{2S},$$

where $S = L_1 \cdot L_2 - M^2$, $Q = r_2 \cdot L_1 + r_1 \cdot L_2$, $R^2 = Q^2 - 4r_1 \cdot r_2 \cdot S$;

$$I_1 = \frac{E_1}{R \cdot r_1}\left[R - \tfrac{1}{2}e^{\lambda t}(r_2 \cdot L_1 - r_1 \cdot L_2 + R) + \tfrac{1}{2}e^{\mu t}(r_2 \cdot L_1 - r_1 \cdot L_2 - R)\right], \quad (22)$$

$$I_2 = \frac{E_1 \cdot M}{R}\left[e^{\mu t} - e^{\lambda t}\right], \quad (23)$$

$$\int_0^\infty I_2 \cdot dt = -\frac{E_1 \cdot M}{r_1 \cdot r_2}, \quad \text{and} \quad \int_0^\infty \left(\frac{E_1}{r_1} - I_1\right) dt = \frac{L_1 \cdot E_1}{r_1^2}. \quad (24)$$

Figure 44 illustrates a typical case where S is positive: the heavy line shows the current in the primary circuit when $r_1 = 3$ ohms, $r_2 = 2$ ohms, $L_1 = 3$ henries, $L_2 = 2$ henries, $M = \sqrt{6/3}$ henries, $E_1 = 12$ volts, when the secondary is closed; the lighter curve shows the rise of the current in the same circuit when the secondary circuit is open.

$$I_1 = 4\,(1 - \tfrac{1}{2}\,e^{-\frac{3t}{4}} - \tfrac{1}{2}\,e^{-\frac{3t}{2}}), \quad (25)$$

and $$I_1 = 4\,(1 - e^{-t}). \quad (26)$$

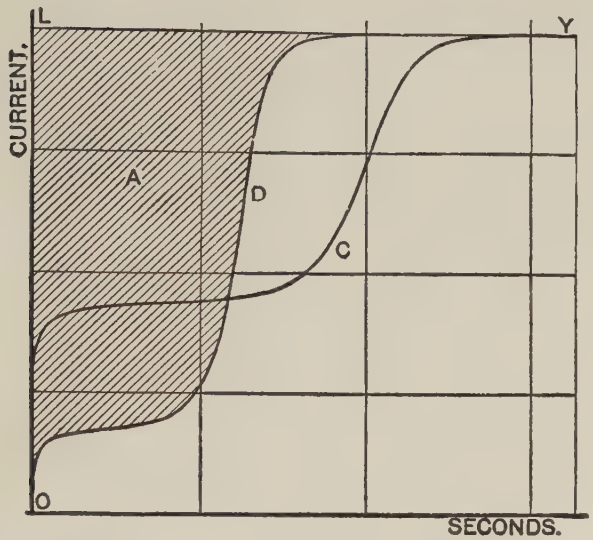


FIGURE 45. — Reverse current curves for the coil of 2788 turns belonging to the magnet Q, when the circuit of a secondary coil of 1095 turns was closed (C) and open (D). The resistance of the primary circuit, which contained a battery of 40 storage cells, was 30 ohms.

The slope of the first curve is at the outset somewhat greater than that of the secondary curve, but eventually becomes less, the curves intersecting at a point Y. The area between the curve and the asymptote drawn parallel to the axis of abscissas is the same for both cases.

If the circuits just described had in common a large closed iron core, the current curves for open and closed secondary circuit would be much less like each other than the curves of Figure 44 are, even if the core were not solid. We may illustrate this fact by some oscillograms from a transformer which has a laminated core.

Figure 45 shows two typical reverse current curves for the exciting coil of the magnet Q which has 2788 turns, when the circuit of a secondary coil of 1095 turns is (D) open and (C) closed. Both curves rise very rapidly at the start, and then bend suddenly, so as to become almost horizontal for a time, but in the first fifth of a second the curve taken when the secondary is closed attains 40 per cent of its final value, and the other curve only 18 per cent; yet the second curve

reaches half its height about two fifths of a second sooner than the first does; and when the secondary is open the current in the primary circuit reaches 98 per cent of its maximum strength in about five-sixths of a second less time than when the secondary is closed. In this case the final current was 2.80 amperes. Of course the degree of divergence of the current curve for the primary circuit when the secondary is closed, from the corresponding curve when the secondary is open, depends very much upon the number of turns of the secondary and upon its resistance.

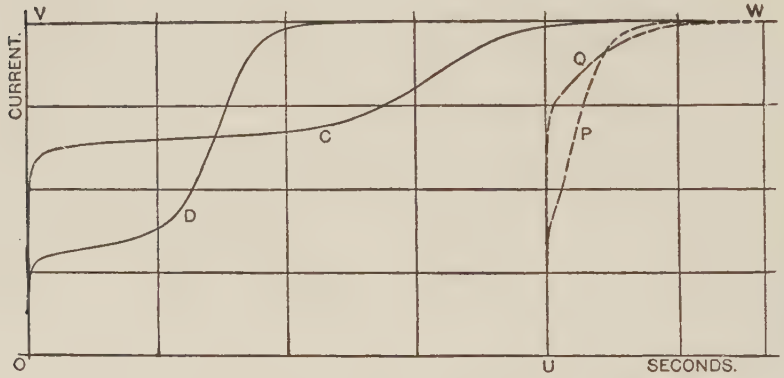


FIGURE 46. — Direct and reverse current curves for a coil of 1394 turns belonging to the magnet Q when a secondary circuit of 1394 turns was closed and open.

Figure 46 shows both reverse and direct curves for the magnet Q when the primary and secondary coils were geometrically alike and each had 1394 turns. The resistance of the primary circuit was about 16.7 ohms.

The curves of Figure 47 belong to a primary coil of 823 turns of the magnet Q. The lines which have O as origin represent currents of about 2.05 amperes due to a storage battery of 10 cells; the lines which start at X were caused by currents of 7.55 amperes from a battery of 40 cells.

Figure 48 shows direct and reverse curves for a current of 3.30 amperes (due to a storage battery of 40 cells) in a coil of 1394 turns belonging to Q. The curves M, N were taken with a secondary coil of 16 turns and comparatively high resistance closed; the boundaries

of the shaded areas m, n show the forms of the currents induced in this secondary as obtained from an oscillograph in the circuit. Since the number of turns in this secondary was so small and the resistance large, the forms of the curves M, N are not very different from what they would have been if the secondary circuit had been open. The

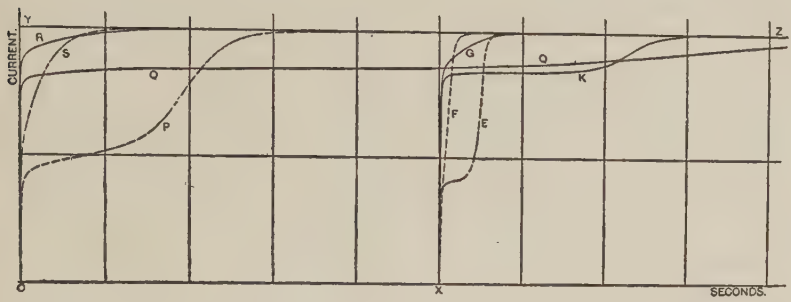


FIGURE 47. — Direct and reverse curves representing currents in a primary coil of 823 turns belonging to the magnet Q, for open and closed secondary circuit. The secondary coil had 2788 turns. For the curves which start at O the voltage was about 20.6; for the curves which begin at X the voltage was about 82 and the maximum current 7.55 amperes.

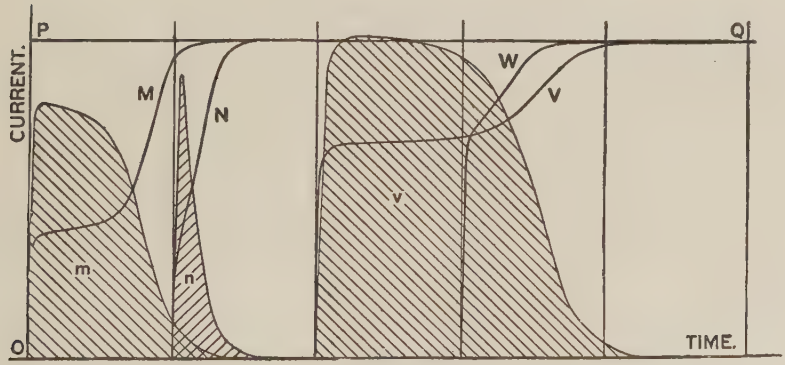


FIGURE 48

curves V, W were taken with another secondary circuit of 1095 turns closed on itself: the boundary of the area v shows on an arbitrary scale the form of the induced current in this last-mentioned secondary circuit.

It is not to be expected, of course, that a current curve for the exciting coil of an electromagnet which has a large solid core will be so

much altered in general appearance by the closing of a secondary coil as it would be if the core were divided so as to prevent in large measure the effects of powerful eddy currents which are present when the iron is in one piece.

Even in the case of an electromagnet the core of which is built up of broad varnished pieces of sheet iron, eddy currents in this iron may

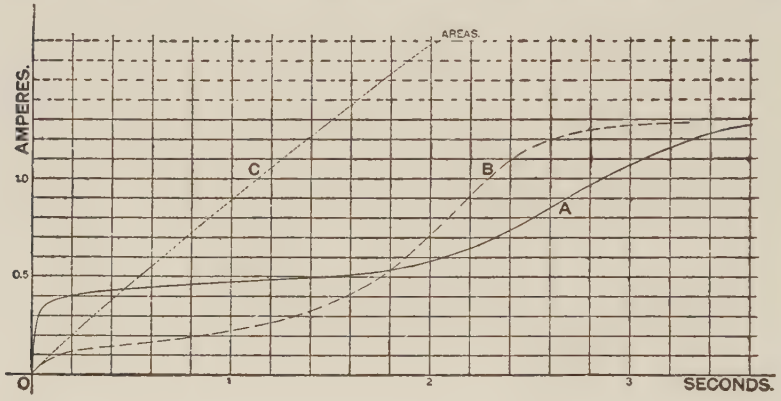


FIGURE 49. — The full line represents the actual form of a reverse current curve in the coil of a certain transformer the core of which is laminated; the curve sketched out by dashes represents the theoretical form as obtained from the statical hysteresis diagram. The dotted curve represents on an arbitrary scale the areas between the real curve and the asymptote; the flux change being nearly proportional to the time.

radically change the form of a current curve unless the sheets are very thin. Figure 49 illustrates this fact by an actual example drawn to scale.

Figure 50 shows curves belonging to a certain transformer. *M* is a piece of a statical hysteresis curve; *N* is a similar curve obtained from a reverse current oscillogram. Although the core of this magnet is made up of varnished pieces of sheet iron, the effects of eddy currents, as will be shown more clearly in the sequel, are here very noticeable.

Some instances of the phenemenon just mentioned suggest a possible pure time-lag [1] of magnetization, like that observed by Ewing and Lord Rayleigh, large enough in the case of a very large core to affect somewhat the forms of the current curves; in fact, I have spent

[1] G. Wiedemann, *Galvanismus*, vol. iii, 738; Ewing, *Magnetic Induction*, § 84; Gumlich und Schmidt, *Electrotechnische Zeitschrift*, vol. xxi, 1900; Rücker, *Inaugural Dissertation*, Halle-Wittenberg, 1905.

a very long time and have made many measurements upon a great number of oscillograph records in order to see whether any such lag could be shown; but after all allowances have been made for the

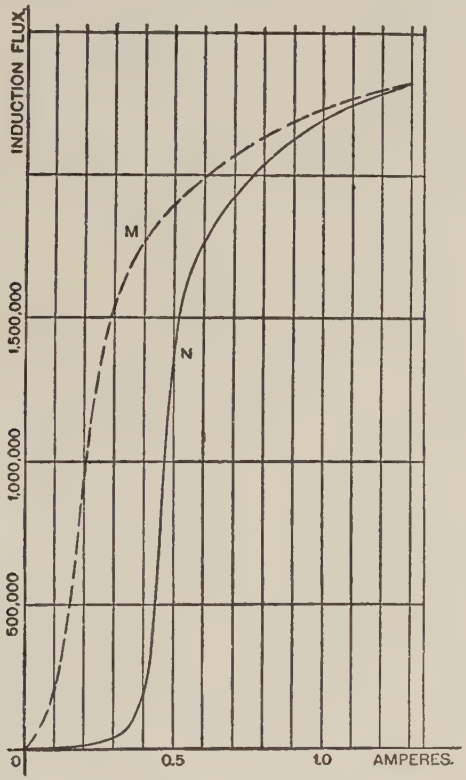


FIGURE 50. — M is a portion of a statical hysteresis diagram for a certain transformer under an excitation of 1812 ampere turns. N is a similar curve obtained from a reverse current oscillogram.

effects of eddy currents, nothing tangible, if anything at all, remains, for such moderate excitations as I have used with closed, finely divided cores.

If to a circuit — without iron and unaffected by any neighboring currents — which has a fixed inductance L, and resistance r, be applied a fixed electromotive force, E, the current-time curve will follow the equation

$$I = \frac{E}{r}(1 - e^{-\frac{rt}{L}}),$$

and the current will attain the intensity $I_0 = E/(r + h)$ at the time t_0 such that

$$e^{-\frac{rt_0}{L}} = \frac{h}{r + h}.$$

If, however, the resistance of the circuit at the outset had been $(r + h)$ and if after the final value of the current I_0 for this resistance had been established, the extra resistance had been suddenly removed from the circuit, the current curve from that instant on would have followed the equation

$$I = I_0 e^{-\frac{rt'}{L}} + \frac{E}{r}(1 - e^{-\frac{rt'}{L}}),$$

or, since

$$I_0 = \frac{E}{r}(1 - e^{-\frac{rt_0}{L}}), \qquad I = \frac{E}{r}(1 - e^{-\frac{r(t' + t_0)}{L}}).$$

It is clear, therefore, that in the case of a circuit of this kind the last (upper) portion of a step curve of the form U (Figure 4) will have exactly the same shape as the corresponding part of the V curve, although the lower portions may be very different.

If in the case also of a circuit which has one or more finely divided iron cores the flux of induction through the circuit can be considered as a single valued (given) function of the current strength when the magnetic state of the iron at the outset is given, the upper portion of a curve of the U type (Figure 4) belonging to the circuit will be identical with the corresponding part of a curve of the V type. We need consider only a U curve with one intermediate step. If the induction (N) through the circuit corresponding to a current of intensity I is $\phi\ (I)$, and if the resistance of the circuit is R, the differential equation which determines the growth of the current is

$$E - \frac{dN}{dt} = RI \quad \text{or} \quad \frac{\phi'\ (I) \cdot dI}{E - RI} = dt.$$

Since ϕ is known, the coefficient of dI is known after values have been assigned to the constants E and R. If with a given E, R has the value r, the curve obtained by plotting the coefficient of dI against I will have a shape something like that of the line $KCDP$ of Figure 51, which has the line $I = E/r$ for an asymptote. If with the same value of the electromotive force R has the value $(r + h)$, the curve will have

a shape something like that of the line $KBDA$, which has the vertical asymptote $I = E/(r + h)$which passes through Q. If with the core in the state for which the diagram is drawn, the circuit be closed at the time $t = 0$, and if the resistance be $(r + h)$, the time required for the current to attain any value I' less than $E/(r + h)$ is proportional to the shaded area under the curve $KBDA$ from the ordinate axis up to the vertical line $x = I'$. If, however, the resistance of the circuit had been r, the time required for the current to grow to the intensity I' would be represented on the same scale by the area under the curve $KCDP$ from $x = 0$, to $x = I'$. If the circuit were closed when its

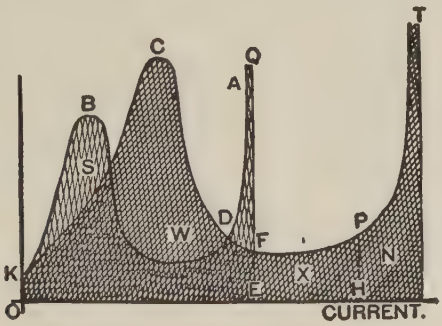


FIGURE 51

resistance was $(r + h)$, and if the current were allowed practically to reach its final value for this resistance, as represented by the line OE, and if then the resistance h were suddenly shunted out, the current would grow to its new final value at a rate determined by the fact that the time required to reach the current OH must be equal, on the scale of the diagram, to the area $EFPH$. If the circuit had been closed first when its resistance was r, the time required for the current to grow from the intensity OE to the intensity OH would still be equal, on the scale used, to the area $EFPH$, and the shape of the current curve, from $E/(r + h)$ on, would be the same as before. Of course the N of this theory need not be the same as the N of the statical hysteresis diagram for the given magnet; it might have for any value of I a value which in the case of the statical curve belonged to a current weaker by any given constant or otherwise determined amount. The curve FP must, however, have the same form for a con-

tinuously growing current and for one which suddenly begins to increase from the value OE.

As a matter of fact, experiment seems to show that if the core of an electromagnet is made of varnished wire so fine that eddy currents are practically shut out, the upper portion of a U curve with a single intermediate step is exactly like the corresponding portion of the V curve. Figure 52 represents a set of current curves obtained from a number of toroidal coils (with very fine wire cores) connected up in series; the

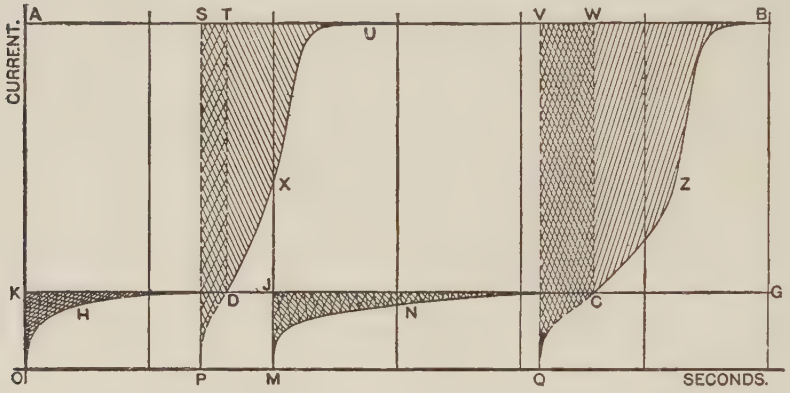


FIGURE 52. — Current curves for a coil with fine wire core. The second part of a two-stage current is exactly the same as if the current were allowed to grow at once to its final value.

current came from a storage battery of ten cells. When the circuit had its normal resistance, the final value of the current was represented by OA; it was possible, however, to close the circuit with such an extra amount of resistance that the final value of the current should be representable on the same scale as before, by the line OK. The extra resistance could then be suddenly shunted out of the circuit by closing a switch at any time after the lower current had practically attained its maximum strength. When the core had been previously demagnetized, a diagram of this kind had the form $OHDXU$; but if the circuit had from first to last its normal resistance, the current curve had a shape accurately represented — when the starting point was shifted to the proper point (P) on the time axis — by $PDXU$. The upper part of the curve was in no way distinguishable from the corresponding portion of the U diagram. Mr. John Coulson and I have taken many

records of this kind and have not been able to detect any difference between the upper parts of the different kinds of curves. The second part of the U diagram starts off at exactly the same angle with the horizontal that the other curve has when the line KG is crossed. The area $OKDHO$ when divided by the length OK should be the same as the area $PSTDP$ divided by the length OA.

If eddy currents are present, the upper portions of a U diagram and of a V diagram do not entirely agree. Figure 53 represents diagrams

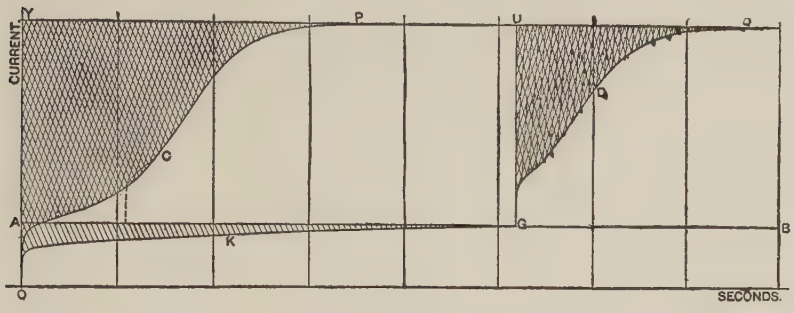


FIGURE 53. — Growth from an originally neutral core of a current in a transformer with a laminated core. The effects of eddy currents are here noticeable.

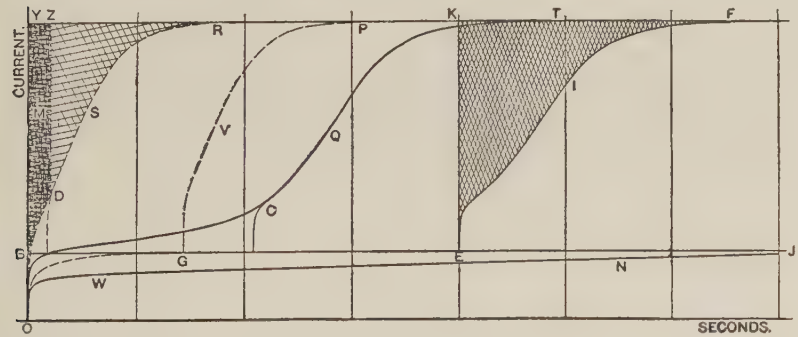


FIGURE 54. — Direct and reverse current curves for a transformer with a laminated core. The existence of eddy currents is clearly shown.

for the magnet Q which has a laminated core, although eddy currents are not entirely shut out. If the upper part of the U diagram (GDQ) be shifted to the left, it will be found to agree with the curve PCO from P to C, but beyond C the two are quite different, as the dotted line indicates. When the V current, the growth of which is represented

by the line *OCP*, has reached the strength *OA*, the induction flux through the core is only a small fraction of the flux when a steady current of final strength *OA* is established in the coil in the manner represented by *OKG*. The existence of eddy currents is indicated clearly by the fact that the first portion of the curve *GDQ* is nearly vertical. These diagrams were obtained when the core had been well demagnetized. Figure 54 shows similar diagrams for direct curves (dotted) and for reverse curves (full).

The Growth of the Induction Flux in the Core of an Electromagnet while the Exciting Current is Temporarily Constant

It sometimes happens that if a number of secondary coils of low resistance, wound upon the core of an electromagnet, are closed on themselves, the building-up curve of a current in the exciting coil is for a comparatively long time almost exactly parallel to the time axis.

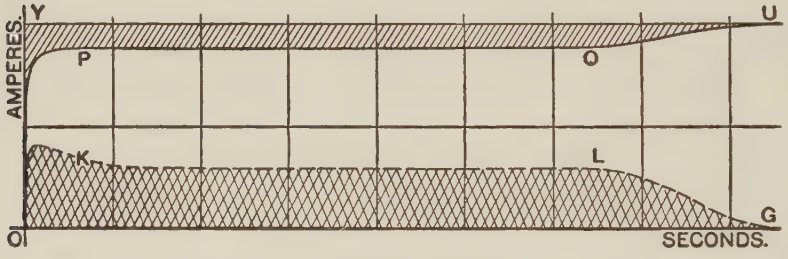


Figure 55

During this time it is difficult to detect any change in the intensity of the current, and yet the flux of magnetic induction through the core is increasing at a very nearly constant rate. This fact, which has a certain pedagogic interest, is easily illustrated. The curve *OPQU* (Figure 55) shows a nearly typical case, and the line *OKLG* represents on a different scale the induced current in one of the secondary circuits. To a person watching an amperemeter in the primary circuit, the current seems to have attained its final value in less than a second, and if he leaves the instrument at the end of, say, five seconds, he feels sure that the current has become steady. Meanwhile the induction flux, as measured on the scale of the diagram by the area be-

tween the curve and the line YU (or, on a different scale, by the area under the curve $OKLG$), is constantly growing. Of course if the core is very large, the whole building-up time may be a minute or more, and the phenomenon may then become very striking.

The magnet T has three coils. The first (A) has 750 turns, the second (B) 250 turns, and the third (C), which is made of wire of very large cross-section, has a small unknown number. Figure 56 reproduces accurately the records of two oscillographs, one in the coil A, the other in B, when C was closed. $OMQL$ is a part of the building-up curve for the main circuit (A), and $Ocbk$ is a corresponding portion of the record of the induced current in B. In the case represented by the full line $OMQTVW$, the coil C was suddenly opened at about 1.05 seconds after the start: $Ocbznda$ shows the record of the induced current in B under these circumstances. The scales of the two oscillographs were, of course, not the same. The sudden jumps in the oscillograms might have been predicted, in amount as well as in direction, by the principle of the "Conservation of Electromagnetic Momenta." We shall return to the subject of the sudden changes brought about in the currents in inductively connected circuits when the inductances of the system are impulsively changed.

The Effectiveness of Fine Subdivision in the Core of an Electromagnet for the Prevention of Electromagnetic Disturbances due to Eddy Currents, when a Steady Electromotive Force is applied to the Circuit of the Exciting Coil

In order to determine approximately the magnitude of the effect of eddy currents upon the growth of a current [1] in the coil of an electromagnet the core of which is made of fine iron wire, we may consider the case of a very long solenoid consisting of N turns of wire per centimeter of its length, wound closely about a long prism of square cross-section ($2a \times 2a$) built up uniformly (Figures 59 and 60) of a large

[1] The influence of eddy currents in the formation of a regularly fluctuating current in the exciting coil of a transformer under a given, alternating electromotive force has been studied by J. J. Thomson for cores of square cross-section built up of iron sheets, and by Heaviside for round cylindrical cores cut radially. See the *Electrician* for April, 1892, and Heaviside's *Electrical Papers*, i, 28.

number of varnished filaments of square cross-section ($c \times c$), or else consisting of a bundle of infinitely long straight wires. The axis of the prism shall be the z axis, and the x and y axes shall be parallel to faces of the prism. The electric resistance of the solenoid per centimeter of its length shall be w, the constant applied electromotive force per centimeter of the length of the prism shall be E, and the intensity of the current in the coil shall be C. Within the core, the magnetic field (H) will have the direction of the z axis, and if q is the current flux at any place

$$4\pi q = \text{Curl } H, \tag{27}$$

or
$$4\pi q_x = \frac{\partial H}{\partial y}, \quad 4\pi q_y = -\frac{\partial H}{\partial x}, \quad 4\pi q_z = 0.$$

Within any filament of iron in the core, H satisfies the equation

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right), \tag{28}$$

where ρ is the specific resistance of the iron and μ is its permeability, which for the present purpose shall be regarded as having a fixed value.

When there are no Foucault currents in the core, the intensity (H) of the magnetic field within has at every point the boundary value H_s or $4\pi NC$, but if positively directed eddy currents exist, H may be greater at inside points than at the surface. We need not distinguish between the flux p through the turns of the coil per centimeter of its length, and N times the induction flux $\mu \iint H\, dx\, dy$ through the core, so that we may write

$$E - \frac{dp}{dt} = E - \mu N \iint \frac{\partial H}{\partial t} \cdot dx\, dy = w \cdot C = \frac{w \cdot H_s}{4\pi N}, \tag{29}$$

or by virtue of (28),

$$E = \frac{w \cdot H_s}{4\pi N} + \frac{\mu\rho N}{4\pi\mu} \iint \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) dx\, dy, \tag{30}$$

where the integration extends over a cross-section of the core.

The vector H is always perpendicular to its curl, and the intensity of the component of the current at any point in the iron, in any direc-

tion, s, parallel to the xy plane at any instant, is equal to $1/4\pi$ times the value at that point, at that instant, of the derivative of H in a direction parallel to the xy plane, and 90° in counter clockwise rotation ahead of s.

Along any curve in the iron parallel to the xy plane, H must be constant if there is no flow of electricity across the curve. At every instant, therefore, the value of H at the boundary common to any two filaments must be everywhere equal to H_s. If the coil circuit is broken, H must be constantly zero at the surface of every filament.

Two or three general theorems concerning solutions of differential equations of the form

$$g\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial w}{\partial z},$$

will be helpful to us.

If v and w represent any analytic functions of x, y, z, and if $L(w)$, $M(v)$ represent the adjoint differential expressions

$$g\cdot\frac{\partial^2 w}{\partial x^2}+g\cdot\frac{\partial^2 w}{\partial y^2}-\frac{\partial w}{\partial z}, \tag{31}$$

$$g\cdot\frac{\partial^2 v}{\partial x^2}+g\cdot\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial z}, \tag{32}$$

the corresponding form of the generalized Green's Theorem may be expressed by the equation,

$$\iiint[v\cdot L(w)-w\cdot M(v)]\cdot dx\,dy\,dz=$$

$$g\iint\left(v\cdot\frac{\partial w}{\partial x}-w\cdot\frac{\partial v}{\partial x}\right)\cdot\cos(x,n)\cdot dS+$$

$$g\iint\left(v\cdot\frac{\partial w}{\partial y}-w\cdot\frac{\partial v}{\partial y}\right)\cdot\cos(y,n)\cdot dS-\iint w\,v\cdot\cos(z,n)\cdot dS; \tag{33}$$

and it is easy to prove that

$$\iiint v\cdot L(w)dx\,dy\,dz=g\iint v\left(\frac{\partial w}{\partial x}\cdot\cos(x,n)+\frac{\partial w}{\partial y}\cdot\cos(y,n)\right)dS$$

$$-g\iiint\left(\frac{\partial w}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial y}\right)dx\,dy\,dz-\iiint\cdot\frac{\partial w}{\partial z}\cdot dx\,dy\,dz. \tag{34}$$

If w and v are identically equal, the last equation becomes

$$\iiint w \cdot L(w) \cdot dx\,dy\,dz = g \iint w \left(\frac{\partial w}{\partial x} \cdot \cos(x, n) + \frac{\partial w}{\partial y} \cdot \cos(y, n) \right) dS$$
$$- g \iiint \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx\,dy\,dz - \tfrac{1}{2} \iint w^2 \cdot \cos(z, n) dS. \quad (35)$$

(I) If S_0 is a closed cylindrical surface the generating lines of which are parallel to the z axis, and if Ω, Ω' — two functions which within S_0 satisfy the equations $L(\Omega) = 0$, $L(\Omega') = 0$ — (1) vanish at all points

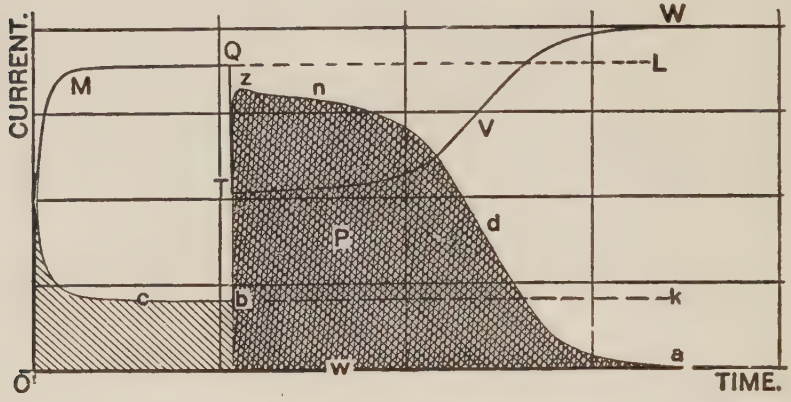


FIGURE 56

of S_0 and at all points within S_0 for which z is positively infinite, and (2) have the given constant value Ω_0 at all points in the xy plane within S_0; then if we apply (35) to the difference between Ω and Ω', using as a field of volume integration the space inside S_0 on the positive side of the xy plane (Figure 57), we shall learn that in this space Ω and Ω' must be identically equal. The value of Ω within S_0 is in no way affected by conditions which a physical extension of the function might be required to satisfy outside S_0.

(II) If S_0 is a closed cylindrical surface, the generating lines of which are parallel to the z axis, if W is a function which within S_0 satisfies the equation $L(W) = 0$, and if

(1) W and $\partial W/\partial z$ vanish at all points within and on S_0 for which z is positively infinite,

(2) W has a given constant value (W_0) at all points on the xy plane within S_0.

(3) W on S_0 is a function (W_s) of z only, such that if n indicates the direction of the external normal to S_0

$$W_S + k \int \left(\frac{\partial W}{\partial n}\right) ds = 0, \tag{36}$$

where k is a given positive constant, and the line integral is to be taken around the perimeter of a right section of S_0 made by the plane $z = z$; and, hence, if

(4) $\iint \left(\frac{\partial W}{\partial z}\right) dS$, taken over so much of the xy plane as lies within S_0, is given, then W is uniquely determined.

If we assume that two different functions (W, W') may satisfy all these conditions, and denote their difference by u,

$L(u) = 0$, within S_0,

u and $\partial u/\partial z$ vanish at all points within S_0, for which z is positively infinite,

u vanishes at all points on the xy plane within S_0,

u on S_0 satisfies the equation

$$u_S + k \int \left(\frac{\partial u}{\partial n}\right) ds = 0. \tag{37}$$

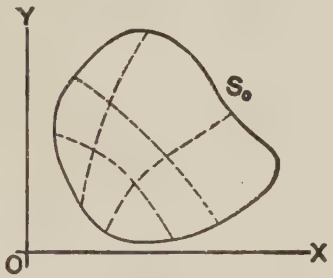


FIGURE 57

If we use the space bounded by S_0, the xy plane, and the plane $z = \infty$, as a field of volume integration, and denote the whole boundary by S, then, since cos (z, n) vanishes on S_0, and u, cos (x, n), cos (y, n), vanish on the portions of the planes $z = 0$, $z = \infty$ used as boundaries (35) yields the equation

$$\iiint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy\, dz = \iint u_S \cdot \frac{\partial u}{\partial n} \cdot dS_0. \tag{38}$$

Now u has the same value at all points on the perimeter (s) of any right section of S_0, so that

$$\iint u_S \cdot \frac{\partial u}{\partial n} \cdot dS_0 = \int_0^\infty u_S \cdot dz \int \frac{\partial u}{\partial n} \cdot ds = -\frac{1}{k} \int_0^\infty u_S^2 \cdot dz, \tag{39}$$

and (38) becomes

$$\iiint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy\,dz + \frac{1}{k}\int_0^\infty u_S{}^2 \cdot dz = 0, \qquad (40)$$

where k is intrinsically positive; but each of these last integrals has an integrand that must be either zero or positive at every point in its domain, so that u must be independent of x and y, and must vanish on S_0 at every point. It follows that u is everywhere zero and that $W = W'$.

It is evident that the condition (3) might have been stated in the form of the equation

$$W_S + k\iint \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) dA = 0, \qquad (41)$$

where the integration is to be extended over so much of the plane $z = z$ as lies within S_0.

If the space within S_0 were cut up into portions (filaments) by the cylindrical surfaces S_1, S_2, S_3 . . ., the generating lines of which were parallel to the z axis, and if within each filament $L(W)$ vanished, while, in addition to the other requirements enumerated above, W were constrained to have at every point of the surface of every filament the value (W_S), which points with the same z coördinate on the surface S_0 had, — though the normal derivative of W at the common surface of two filaments were not expected to be continuous, — we might assume as before that two different functions could satisfy all these conditions and denote their difference by u. We could then apply (35) to every filament separately (Figures 57 and 58) and obtain from each an equation of the form

FIGURE 58

$$\int u_S \cdot dz \iint \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dB - \iiint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dz\,dB = 0, \qquad (42)$$

where B denotes a cross-section of the filament. If, then, all these equations were added together, the resulting equation would be

$$\int u_S \cdot dz \iint \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dA - \iiint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dz\,dA = 0, \qquad (43)$$

which is (35). In this case also, therefore, W is determined.

(III) If S_0 is a closed cylindrical surface the generating lines of which are parallel to the z axis, if V is a function which within S_0 satisfies the equation $L(V) = 0$, and if

(1) V and $\partial V/\partial z$ vanish at all points within and on S_0 for which z is positively infinite,

(2) V has a given constant value (V_0) at all points on the xy plane within S_0,

(3) V on S_0 is a function (V_S) of z only, such that, if n indicates the direction of the external normal to S_0

$$V_S + l \cdot \frac{dV_S}{dz} + k \int \left(\frac{\partial V}{\partial n}\right) ds = 0,$$

or

$$V_S + l \cdot \frac{dV_S}{dz} + k \iint \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) dx\, dy = 0, \tag{44}$$

where l and k are given positive constants, the line integral is to be taken around the perimeter (s) of a right section of S_0 made by the plane $z = z$, and the double integral over the section; then V is uniquely determined.

(IV) Let S_0 be a closed cylindrical surface which completely surrounds (Figure 58) several other mutually exclusive, closed cylindrical surfaces (S_1, S_2, S_3, . . .) the generating lines of which are parallel to those of S_0 and to the z axis; and let the intersections of these surfaces with the plane $z = z$ be denoted by s_0, s_1, s_2, s_3, ... Let the portions of the plane $z = z$ within S_1, S_2, S_3, . . ., be denoted by A_1, A_2, A_3, . . ., and the portion within S_0 but outside S_1, S_2, S_3, . . ., be denoted by A_0. Let τ_0, τ_1, τ_2, τ_3, . . ., represent the volumes of the prisms (bounded by the planes $z = 0$, $z = \infty$) of which the cross-sections made by the planes $z = z$ are A_0, A_1, A_2, A_3, ...

In the regions τ_0, τ_1, τ_2, τ_3, . . ., let the scalar function U satisfy the equations

$$\begin{aligned}
\frac{\partial U}{\partial z} &= g_0\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \\
\frac{\partial U}{\partial z} &= g_1\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \\
\frac{\partial U}{\partial z} &= g_2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{45}$$

where g_0, g_1, g_2, g_3 are given positive constants, and let the value (U_S) of U on the cylindrical surfaces be a function of z only (the same for all the surfaces), such that

$$U_S + k_0 \iint \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_0 + k_1 \iint \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_1 + k_2 \iint \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_2 + \ldots = 0, \tag{46}$$

where k_0, k_1, k_2, k_3 are given positive constants. Then if U has the constant value U_0 at all points in so much of the xy plane as lies within S_0 and the value zero at all points on and within S_0 for which z is positively infinite, U is determined in the positive space within S_0. For if we assume that there could be two such functions and apply (35) to their difference (u) in each of the regions τ_0, τ_1, τ_2, τ_3, . . ., multiply the resultant equations by k_0, k_1, k_2, k_3, . . ., and add them together, it will be easy to show — in the way indicated under (II) — that u is zero everywhere inside S_0 on the positive side of the xy plane.

It is to be remembered that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \tag{47}$$

is an invariant of a transformation of orthogonal Cartesian coördinates in the xy plane.

(V) In an important special case similar to that stated in (IV), k_1, k_2, k_3, . . ., are all equal, g_1, g_2, g_3, . . ., are all equal, and all the n^2 areas A_1, A_2, A_3, . . ., are alike in form, however they may be oriented. In the region τ_0, U is everywhere equal to U_S, which is, as before, a function of z only, and the surface condition becomes

$$U_S + l \cdot \frac{dU_S}{dz} + k \sum_m \iint \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) dA_m, \tag{48}$$

where l and k are given positive constants.

If in this case we find for every one (τ_m) of the regions τ_1, τ_2, τ_3, . . ., the function (w_m), which within (τ_m) satisfies the equation

$$\frac{\partial w_m}{\partial z} = g \left(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \right), \tag{49}$$

and at the boundary the surface condition

$$w_S + l \cdot \frac{dw_S}{dt} + n^2 k \iint \left(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} \right) dA_m = 0, \qquad (50)$$

and which has the given constant value U_0 on so much of the xy plane as lies within S_0 and the value zero when z is infinite, and if we assign to the function without S_m where it is not defined, the value zero, then, apart from differences of orientation, all these functions will be alike. If after this we define a function within S_0 by assigning to it within every one of the regions $\tau_1, \tau_2, \tau_3, \ldots$, the same value as the w function belonging to this region, and give to it in τ_0 the common value w_s, the function thus determined will be the unique function U described above.

If after a steady current of intensity E/w has been running for some time in the coil of the solenoid under consideration, so that the magnetic field within the core (which in this case shall be built up, in the manner shown in Figure 59, of filaments of square cross-sections) has everywhere the given constant value H_0, the coil circuit be very suddenly broken, the value of H falls instantly, not only at the outer surface of the prism, but also at the surface of every filament, to zero. Inside every filament

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu} \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right). \qquad (51)$$

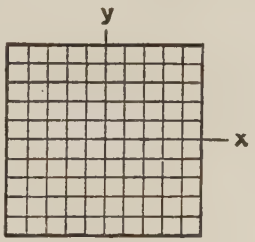


FIGURE 59

When $t = 0$, $H = H_0$ everywhere within the iron, and when t is infinite, the field intensity is everywhere zero. According to (I), therefore, we may consider every filament by itself.

If we seek a solution of the equation (51) which shall be of the form $X \cdot Y \cdot T$, where X involves x alone, Y involves y alone, and T is a function of t alone, we shall obtain the expressions

$$X = A_1 \cdot \cos \alpha x + A_2 \cdot \sin \alpha x, \quad Y = B_1 \cdot \cos \beta y + B_2 \cdot \sin \beta y, \quad T = e^{-\lambda^2 t}, \qquad (52)$$

where

$$\lambda^2 = \frac{\rho(\alpha^2 + \beta^2)}{4\pi\mu}. \qquad (53)$$

If we use as normal function the product

$$A_{mn} \cdot e^{-\lambda^2 t} \cdot \sin \frac{m\pi x}{c} \cdot \sin \frac{n\pi y}{c}, \tag{54}$$

where $\lambda^2 = \pi\rho(m^2 + n^2)\,(4\mu c^2)$ and m and n are positive integers, and write

$$H = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn} \cdot e^{-\lambda^2 t} \cdot \sin \frac{m\pi x}{c} \cdot \sin \frac{n\pi y}{c}, \tag{55}$$

this expression will satisfy all conditions if A_{mn} be so taken that when $t = 0$, the second number of the equation shall be equal to H_0 for all values of x and y within the filament. We have, therefore, the equation [1]

$$A_{mn} = \frac{4H_0}{c^2} \int_0^c d\chi \int_0^c \sin \frac{m\pi\chi}{c} \cdot \sin \frac{n\pi\psi}{c}\, d\psi \tag{56}$$

and

$$A_{mn} = \frac{16}{\pi^2} \frac{H_0}{mn},$$

when m and n are both odd;

$$A_{mn} = 0,$$

when either m or n is even, so that

$$H = \frac{16H_0}{\pi^2} \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=\infty} \frac{e^{-\lambda^2 t}}{(2j+1)\,(2k+1)} \cdot \sin \frac{(2k+1)\pi x}{c} \cdot \sin \frac{(2j+1)\pi y}{c} \tag{57}$$

$$\lambda^2 = \frac{\pi\rho}{4\mu c^2}[(2k+1)^2 + (2j+1)^2]. \tag{58}$$

From (58) it appears that the whole flux of magnetic induction through the core at the time t is

$$\phi = \frac{64 \cdot \mu \cdot H_0 \cdot c^2}{\pi^4} \sum_{j=1}^{j=\infty} \sum_{k=1}^{k=\infty} \frac{e^{-\lambda^2 t}}{(2j+1)^2(2k+1)^2}, \tag{59}$$

or, if

$$g = \pi\rho/4\mu c^2,$$

$$\phi = \frac{64 \cdot \mu \cdot H_0 \cdot c^2}{\pi^4} \sum_{j=1}^{j=\infty} \frac{e^{-g(2j+1)^2 t}}{(2j+1)^2} \sum_{k=1}^{k=\infty} \frac{e^{-g(2k+1)^2 t}}{(2k+1)^2} \tag{60}$$

[1] Byerly, *Treatise on Fourier's Series*, etc., § 71; Riemann-Weber, *Die partiellen Differential-gleichungen der mathematischen Physik*, Bd. II, § 99.

In these equations absolute electromagnetic units are to be used, and for good soft iron we may assume that $\pi\rho/4$ is very approximately equal to 8000. It is evident that for different values of c when μ is given, $e^{-\lambda^2 t}$ will have the same numerical value for values of t proportional to c^2; for instance, if $c = 20$, $t = 10$, $e^{-\lambda^2 t}$ will have the same value as it would if c were 1 and t, 1/40. If c is fixed, $e^{-\lambda^2 t}$ will have the same value for values of t proportional to μ.

It is possible to show that if $c = 1$ and $\mu = 200$, — to take a special case, — the series

$$S \equiv \sum_{k=0}^{k=\infty} \frac{e^{-g(2k+1)^2 t}}{(2k+1)^2} \tag{61}$$

has at different times the approximate values given in the following table:

TABLE V

t	S	t	S
0	1.2337	0.01000	0.6734
0.00025	1.1450	0.02000	0.4494
0.00050	1.1084	0.02500	0.3679
0.00100	1.0565	0.05000	0.1353
0.00200	0.9830	0.07500	0.04979
0.00250	0.9534	0.10000	0.01832
0.00500	0.8374	0.20000	0.00034

From the numbers in this table it is easy to compute, for cores of square cross-section, the fractional part of the original induction flux through the core which remains after the circuit of the exciting coil has been broken for a given time. For a solid core, the area of the

TABLE VI

Time in Seconds after the Breaking of the Circuit	Fractional Part of Original Flux still remaining	Time in seconds after the Breaking of the Circuit	Fractional Part of Original Flux still remaining
0.000	1.000	1.000	0.298
0.025	0.861	2.000	0.133
0.050	0.807	2.500	0.089
0.100	0.733	5.000	0.012
0.200	0.635	7.500	0.0016
0.250	0.597	10.000	0.0002
0.500	0.461		

square section of which is 100 square centimeters, the results are given in the next table, when μ is 200.

If the core were built up compactly of varnished square rods of one square centimeter in cross-section, the times in the table should be

divided by 100, and if the core were made up of 10,000 slender filaments, the flux would sensibly disappear during the first thousandth of a second. It is easy to get similar results for any other value of μ.

If the cross-section of the core were a circle of radius a, and if after a uniform magnetic field of strength H_0 had been established in the core the exciting circuit were suddenly broken, the intensity of the field at any time, at any point distant r centimeters from the axis, would be given by the expression [1]

$$H = \frac{2H_0}{a} \cdot \sum_k \frac{J_0(n_k \cdot r)}{n_k \cdot J_1(n_k \cdot a)} e^{-\beta_k^2 t} \tag{62}$$

where $\beta^2 = \rho n^2/4\pi\mu$ and the whole flux through the core would be

$$2\pi\mu \int_0^a Hrdr \text{ or } 4\pi\mu H_0 \sum_k \frac{e^{-\beta_k^2 t}}{n_k^2} \cdot \tag{63}$$

In these equations $n_k a$ is the kth root in order of magnitude of the Bessel's Equation

$$J_0(na) = 0. \tag{64}$$

The first ten roots are as follows:

TABLE VII

k	na	k	na
1	2.404826	6	18.071064
2	5.520078	7	21.211637
3	8.653728	8	24.352472
4	11.791534	9	27.493479
5	14.930918	10	30.634606

From these numbers the β's can be found, and then from (63) the flux in the core after any interval. When the time is short, the series converges very slowly, and the computation is long and troublesome, but for relatively large values of t the work is not difficult.

The next table shows the fractional part (Ω) of the original flux remaining in a core, the cross-section of which is a circle of 20 centimeters diameter, and in which μ is 200; 1 second, 4 seconds, and 8 seconds after the breaking of the exciting circuit: the corresponding fraction for a core of square cross-section (20 cms. $\times$ 20 cms.) is given

[1] Heaviside, *Electrical Papers*, vol. i, 28; Peirce, *Proceedings of the American Academy*, vol. xli, 1906; Byerly, *Treatise on Fourier's Series*, etc., p. 229.

for comparison. The actual value of the original flux is of course a little larger in the second case because the area of the cross-section is greater.

TABLE VIII

t	Ω for the Round Core	Ω for the Square Core
1	0.588	0.597
4	0.270	0.298
8	0.106	0.133

After 16 seconds Ω for the round core would be 0.016. In the case of a round core of exactly the same cross-section area as the square solid core, and the same original flux, the fractional part remaining after one second would be 0.630.

If the square core of the solenoid — the area of the cross-section of which is A square centimeters — be made of a bundle of infinitely long, straight iron wires, placed close together (Figure 60), and if, after a steady current of intensity E/w has been running for some time through the solenoid, so that there is a magnetic field of uniform intensity $H_0 = 4\pi NE/w$ in the core, the applied electromotive force be suddenly shunted out of the solenoid circuit, the current (C) in the coil will gradually die out. At any instant the field, in so much of the space A as is occupied by air, is $4\pi NC$, for eddy currents in the wires act like solenoid sheets and do not affect the field without the wires. Within each wire there are eddy currents, of course, and at every point in the wire, at every instant, the field intensity, H, must satisfy the equation

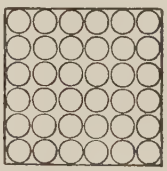


FIGURE 60

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right]. \tag{65}$$

The induction flux through the turns of the solenoid per centimeter of its length shall be p, so that

$$E - \frac{dp}{dt} = wC, \quad \text{or, in this case,} \quad \frac{dp}{dt} = -wC.$$

If there are n^2 wires in the core and the area of the cross-section of each of them is B,

$$p = 4\pi N^2 C(A - n^2 B) + \mu N \iint H \cdot dx\,dy \tag{66}$$

where the double integral is to be extended over the cross-sections of all the wires; hence

$$wC + (A - n^2 B)4\pi N^2 \cdot \frac{dC}{dt} + \mu N \iint \frac{\partial H}{\partial t} \cdot dx\,dy = 0; \tag{67}$$

and if the wires fill the square space as full as possible,

$$A - n^2 B = 0.2146\,A, \text{ nearly.}$$

If H_S represents the strength of the magnetic field in the air space within the solenoid,

$$H_S + \frac{4\pi N^2}{w}(A - n^2 B)\frac{dH_S}{dt} + \frac{4\pi\mu N^2}{w} \iint \frac{\partial H}{\partial t} \cdot dx\,dy = 0. \tag{68}$$

The function H thus defined falls under theorem (V) above, and it is evident that we ought to seek, for a single wire, a function $\bar{\omega}$ which within the wire shall satisfy (65), at the surface shall fulfill the condition

$$\bar{\omega}_S + \frac{4\pi N^2}{w}(A - n^2 B)\frac{d\bar{\omega}_S}{dt} + \frac{4\pi\mu n^2 N^2}{w} \iint \frac{\partial \bar{\omega}}{dt} \cdot dx\,dy = 0, \tag{69}$$

and which when $t = 0$ shall have the value H_0 and when t is infinite, the value zero. When we have to deal with a single wire of radius $b(=a/n)$ alone, it is obviously convenient to use polar coördinates with origin at the point where the axis of the wire cuts the xy plane, and if we do this (65) and (67) take the forms

$$\frac{\partial \bar{\omega}}{\partial t} = \frac{\rho}{4\pi\mu r} \cdot \frac{\partial}{\partial r}\left[r \cdot \frac{\partial \bar{\omega}}{\partial r}\right], \tag{70}$$

$$\bar{\omega}_S + \frac{4\pi N^2}{w}(A - n^2 B)\frac{d\bar{\omega}_S}{dt} + \frac{2\pi n^2 N^2 \rho b}{w}\left(\frac{\partial \bar{\omega}}{\partial r}\right)_{r=b} = 0, \tag{71}$$

or

$$\bar{\omega}_S + l \cdot \frac{d\bar{\omega}_S}{dt} + k n^2 b\left(\frac{\partial \bar{\omega}}{\partial r}\right)_{r=b} = 0, \tag{72}$$

where l, k, n, and b are given, positive constants.

If we attempt to find a solution of (70) in the form of the product of a function of t, and a function of r, we arrive, of course, at the normal form

$$e^{-\beta^2 t}[L \cdot J_0(mr) + M \cdot K_0(mr)], \tag{73}$$

but Bessel's Functions of the second kind will not be needed here, and we may write, $M = 0$,

$$\bar{\omega} = \sum_m L_m \cdot e^{-\beta^2 t} \cdot J_0(mr), \tag{74}$$

where either m or β may be assumed at pleasure and the other computed from the equation

$$m^2\rho = 4\pi\mu\beta^2. \tag{75}$$

If for m in the equation (74) we use the successive roots of the transcendental equation

$$J_0(mb) = \frac{kn^2 \cdot mb}{1 - l\beta^2} \cdot J_1(mb) \tag{76}$$

the series will satisfy (70) and (72), and if the coefficients can be so chosen as to make

$$\sum_0^\infty L_m \cdot J_0(mr) = H_0 \tag{77}$$

equation (74) will give the function sought.

Although the development (77) is not one of those for which the coefficients can be found by the usual devices, it is easy to solve the problem, for such cases as are of practical interest, to any desirable approximation.

We shall find it instructive, however, to inquire first what the solution would be if the second term of (72) were lacking, for, in view of the fact that the permeability of the iron is relatively large compared with that of the air, it seems likely that in some instances, where the series is very convergent, this modified problem and the real one will have nearly equal numerical answers.

We have, then, so to choose L_m, β, and m, subject to (75) that the value of the series (77) shall be H_0 when $t = 0$, for all values of r up to b; and that at every instant

$$\bar{\omega}_S + \frac{2\pi n^2 N^2 \rho b}{w}\left(\frac{\partial\bar{\omega}}{\partial r}\right)_{r=b} = 0. \tag{78}$$

It is necessary, therefore, that m shall be a root of the transcendental equation

$$J_0(mb) = \frac{2\pi N^2 n^2 \rho}{w} \cdot mb \cdot J_1(mb), \tag{79}$$

which may be written in other forms by virtue of the relations

$$\frac{dJ_0(x)}{dx} = -J_1(x), \qquad \int_0^x x \cdot J_0(x)dx = x \cdot J_1(x). \tag{80}$$

It will be convenient to illustrate the effect of making b small (and therefore n large) while a is kept constant, by a numerical example. Let us assume that the cross-section of the solenoid is a square of 10 centimeters side-length, so that $a = 5$; let the solenoid have 10 turns of insulated wire per centimeter of its length, and let the resistance of these 10 turns be $\frac{1}{16}$th of an ohm, so that in absolute units $w = 10^9/16$. If, then, we take the specific resistance of the core to be $(10^6/32\pi)$ absohms at the room temperature (Fleming and Dewar), $2\pi N^2\rho/w$ will be equal to $\frac{1}{10}$, and the equation for m takes the form

$$J_0(mb) = \frac{n^2}{10}(mb) \cdot J_1(mb) = \frac{mb}{\lambda} \cdot J_1(mb). \tag{81}$$

But [1]
$$1 = \sum \frac{2\lambda \cdot J_0(mr)}{(\lambda^2 + m^2b^2)J_0(mb)}, \tag{82}$$

and hence
$$\bar{\omega} = 2\lambda H_0 \sum_m \frac{e^{-\beta^2 t} \cdot J_0(mr)}{(\lambda^2 + m^2b^2)J_0(mb)}. \tag{83}$$

The whole flux of magnetic induction through the iron of the core is then μn^2 times the integral of $\bar{\omega}$ taken over the circle of radius b in which $\bar{\omega}$ is defined; that is

$$\phi = 4\pi\mu\lambda H_0 n^2 b \sum \frac{e^{-\beta^2 t} \cdot J_1(mb)}{m(\lambda^2 + m^2b^2)J_0(mb)}, \tag{84}$$

or
$$\phi = 4\pi\mu\lambda^2 H_0 n^2 \sum \frac{e^{-\beta^2 t}}{m^2(\lambda^2 + m^2b^2)}. \tag{85}$$

Since $\lambda = 10/n^2$, the coefficient of the series may be written 400 $\pi\mu H_0/n^2$, and we may assume that $\mu = 100$.

The time rate of change of the total induction flux through the turns of the solenoid, per centimeter of its length, is

$$\frac{9950 \cdot 10^4 \cdot H_0}{n^2} \sum \frac{e^{-\beta^2 t}}{\lambda^2 + m^2b^2}. \tag{86}$$

If the square core is built up of 100 circular rods, each 1 centi-

[1] Byerly, *Treatise on Fourier's Series*, etc., p. 229.

meter in diameter, $n^2 = 100$, $\lambda = 1/10$, and the m's are defined by the equation

$$J_0(mb) = 10\, mb \cdot J_1(mb) \tag{87}$$

in which $b = 1/2$.

It is not difficult to show by trial and error from Meissel's tables [1] that the first five roots of this equation have values approximately equal to those given in the following table:

TABLE IX

$m_1 b = 0.44168$	$\log \beta_1^2 = 0.79077$	$m_1^2 = 0.78032$
$m_2 b = 3.858$	$\log \beta_2^2 = 2.6733$	$m_2^2 = 59.527$
$m_3 b = 7.030$	$\log \beta_3^2 = 3.1946$	$m_3^2 = 197.672$
$m_4 b = 10.183$	$\log \beta_4^2 = 3.5164$	$m_4^2 = 414.798$
$m_5 b = 13.331$	$\log \beta_5^2 = 3.7504$	$m_5^2 = 710.884$

A mere inspection of these values shows that the value of ϕ can be computed with an accuracy much more than sufficient for any practical purpose from the first two terms of the series (85), if t is as great as $\frac{1}{100}$th of a second, and from the first term alone if t is as great as $\frac{1}{40}$th of a second. Let ϕ_0 represent the first term of (85), then

$$\phi_0 = \frac{400\pi H_0 e^{-6.1768t}}{(0.78032)(0.20508)},$$

but

$$\frac{400}{(0.78032)(0.20508)} = 2499.55, \tag{88}$$

which differs from 2500 by about $\frac{1}{50}$th of one per cent only.

If there were no eddy currents in the iron, the total induction flux through the rods which make up the core would be

$$\phi' = \pi\mu a^2 H'_S, \tag{89}$$

and if C' were the strength of the current in the exciting coil at the time t, we should have

$$\pi\mu a^2 N \cdot \frac{dH'_S}{dt} = -w \cdot C' = \frac{-w \cdot H'_S}{4\pi N} \tag{90}$$

and

$$H'_S = H_0 e^{-ht}, \tag{91}$$

where

$$h = w/4\pi^2 N^2 a^2 \mu = 6.332573 +$$

and

$$\phi' = \pi\mu a^2 H_0 e^{-ht}. \tag{92}$$

[1] Meissel, "Tafel der Bessels'schen Functionen," *Berliner Abhandlungen*, 1888; Gray and Mathews, *Treatise on Bessel's Functions*, pp. 247–266; Peirce and Willson, *Bulletin of the American Mathematical Society*, 1897.

In the case under consideration we should have very nearly

$$\phi' = 2500\pi H_0 e^{-6.332573t} \tag{93}$$

$$4\pi NC' = H'_S = H_0 e^{-6.332573t}. \tag{94}$$

When there are eddy currents the value of H_S is given with sufficient accuracy by the first term of (83) very soon after the electromotive force has been shunted out of the circuit, that is by the equation,

$$H_S = \frac{2000}{2051} \cdot H_0 e^{-6.1768t} \tag{95}$$

and the ratio of ϕ to $\pi b^2 n^2 \mu H_S$ is practically equal to the constant 2051/2000, for it is easy to find a very convergent geometrical series every term of which is greater than the corresponding term of the series which begins with the second term of (85), and the sum of this geometrical series is extremely small except for very small values of t.

According to this analysis, the current in the solenoid will have fallen in the first second to the fraction 0.002025 or to the fraction 0.001777 of its original value according as there are or are not eddy currents in the iron.

If the ten centimeter square iron core of the solenoid were built up of straight rods only one millimeter in diameter, we should have $b = 1/20$, $n = 100$, and $\lambda = 1/1000$; the m's would need to be roots of the equation

$$J_0(mb) = 1000\, mb \cdot J_1(mb). \tag{96}$$

By using differences of the third order it is possible to show from Meissel's table that the first root is approximately equal to 0.044715 + and the second to 3.83. For the first, then, $\lambda^2 + m^2 b^2 = 0.002000$, and $\beta^2 = 6.33077$. For the second root, $\beta^2 = 46500$, and the second terms of the series (83) and (85) become negligible almost immediately after the electromotive force has been removed from the circuit.

In this case

$$\phi_0 = 2500\pi H_0 \cdot e^{-6.33077t} \tag{97}$$

very nearly; and

$$\frac{C}{4\pi N} = H_S = H_0 \cdot e^{-6.33077t}, \tag{98}$$

so that the disturbing effects of the eddy currents are comparatively slight. At the end of one second, the current will have fallen to the

fraction 0.001777 of its original value or to the fraction 0.001781, according as eddy currents were absent or existent. These differ by only about one two hundred and fifty thousandth part of the original current strength. We may note in passing that a very approximate value (correct to four significant figures) of the first root of the equation might be found by equating to unity the coefficient of the first term of the series (83).

If the core of the solenoid were made of wire one tenth of a millimeter in diameter, such as is now in common use in coils intended for loading long telephone circuits, we should have $b = 1/200$, $n = 1000$, $\lambda = 1/100000$, and m would need to satisfy the equation

$$J_0(mb) = 100000\, mb \cdot J_1(mb) \tag{99}$$

It is easy to see that the first root of this has a value very nearly equal to 0.0044721, and that the effects of eddy currents would be quite inappreciable.

Having considered somewhat at length — on the supposition that the induction flux in the air spaces of the core might be neglected — the manner in which a current in the solenoid would decay if the electromotive force were suddenly removed from the circuit without changing the resistance, we may now return to the more general case to which the equations (74) and (76) belong, and remark that in the ideal case where eddy currents are supposed to be absent (68) takes the form

$$H'_S + \frac{4\pi N^2}{w}(21.46)\frac{dH'_S}{dt} + \frac{4\pi\mu N^2 n^2 \pi b^2}{w} \cdot \frac{dH_S}{dt} = 0, \tag{100}$$

whence

$$H'_S = H_0 \cdot e^{-6.31567t}. \tag{101}$$

It is clear at the outset that the larger roots, at least, of the two equations (76) and (79) will be very different, since the second member of (76) soon has a negative coefficient. If then the coefficients of the series (77) could be found, the series (74) and (83) would not resemble each other in appearance for large values of b and small values of the time. If, however, b is fairly small, as it usually is in practice, we may dismiss all thought of the infinite series, since it is easy to choose the coefficients of two or three terms of the form (73) so that

the initial condition shall be satisfied very approximately. In many cases one term suffices.

Let us consider first the case — already treated in another way — of a square core of 100 square centimeters cross-section, built up of long straight wires 1 millimeter in diameter; so that $b = 1/20$, $n = 100$, $l\beta^2 = 1.36620\ m^2b^2$, $kn^2 = 1000$, and the equation for mb has the form

$$J_0(x) = \frac{1000x}{1 - 1.36620x^2} J_1(x). \tag{102}$$

It is possible to show by a rather long application of the method of trial and error, using third differences in Meissel's table, that the value of the first root is 0.044654 + and this corresponds to $m = 0.89308$, $\beta^2 = 6.31351$, $J_0(mb) = 0.9994891+$.

If, then, we consider the single term

$$Q = H_0 e^{-6.31351t} \cdot J_0(0.89308r), \tag{103}$$

Q will satisfy (70) and will vanish when t is infinite. When t is zero, Q will be equal to H_0 for $r = 0$, and will differ from H_0 by about one twentieth of one per cent when $r = b$. The second root of (102) is roughly equal to 3.8 and the corresponding value of β^2 is about 45,000, so that the exponential factor would soon be very small. An inspection of the graph of $J_0(x)$ shows that if we were to use several terms of the form $L \cdot e^{-\beta^2 t} \cdot J_0(mr)$, we could easily form a function which should differ very little from H_0 for any value of r up to b, when t was zero; but it is clear that after the lapse of about 1/5000th of a second, all the terms beyond the first would be negligible, and there is no practical advantage in using more than one term.

We may assume then that the value of H in any one of the iron rods is given fairly accurately, except at the very beginning, by (103). Since $4\pi NC = H_S$ the current in the solenoid falls in the first second to 0.001808 of its original value, or to 0.001812 times that value according as eddy currents are absent or present. These fractions differ from each other by about one two hundred and fifty thousandth part of the original current strength. Another close approximation to the value of H may be made by dividing (103) by $J_0(mb)$ and another by multiplying the second member of (103) by

$$\frac{1 + J_0(mb)}{2J_0(mb)}. \tag{104}$$

These changes would not affect the relative rate of decay of the current.

The nearness of the approximation to the value of the field attainable by a single term is evidently much increased as the diameter of the iron wire of which the core is built up is decreased. If as before $a = 5$, but if $b = 1/200$, $n = 1000$, the value of the first root of the equation for mb will be 0.00446616, nearly, and the value of $J_0(mr)$ will not change by so much as 1/100000th part of itself as r changes from 0 to b. A single term, therefore, will represent H with great accuracy. In this case the effect of eddy currents is wholly inappreciable. Of course this statement does not apply to the case of an alternate current of very great frequency.

In the problem just considered the electromotive force was suddenly shunted out of the solenoid circuit after a steady current had been established in it, and, on the assumption that the permeability of the iron was fixed, the value of the magnetic field within the core was determined as a function $[H_0 f(t, r)]$ of the time and the space coordinates. The function f satisfies (65) and (68), vanishes when t is infinite, and is initially equal to unity. If the solenoid circuit containing an applied electromotive force E be suddenly closed at the time $t = 0$, and if the ultimate value $(4\pi NE/w)$ of the magnetic field in the core be denoted by H_∞, the value of the field at any time will be given by the equation

$$H = H_\infty [1 - f(t, r)]. \tag{105}$$

The function defined by this equation vanishes, when $t = 0$, for all values of r, and when t is infinite is equal to H_∞. It satisfies at all times the equation (65) and the surface equation

$$H_S + \frac{4\pi N^2}{w}(A - n^2 B)\frac{dH_S}{dt} + \frac{4\pi\mu N^2}{w}\iint \frac{\partial H}{\partial t}\, dx\, dy = \frac{4\pi N}{w} \cdot E, \tag{106}$$

and such a function is evidently unique.

Although in practice the permeability is not fixed, the analysis of this section enables us to shut in between narrow limits the effects of eddy currents in many cases, and to assert, when this is the truth, that in a given instance the effects of such currents will be negligible, if the pieces of which the core is built are properly varnished.

It is sometimes possible to get interesting information about the magnetic properties of the core of a transformer which has several coils, and about the excellence of the insulation of the sheets of which it is made, by observing the sudden changes in the currents in the coils when the inductances of the system are impulsively changed, or by studying the rate of propagation of the induction flux into the core, but these subjects must be left for the next instalment of this paper.

IX

THE DAMPING OF THE OSCILLATIONS OF SWINGING BODIES BY THE RESISTANCE OF THE AIR [1]

When a body, free to turn about a fixed axis, like a horizontal pendulum, a suspended magnet, or the coil of a d'Arsonval galvanometer, is disturbed from a position of equilibrium, and is then allowed to swing under the action of a righting moment the intensity of which is proportional to the angular deviation of the body from the position of rest which it originally had, the damping effect of the resistance which the air offers to the motion is sooner or later made evident by a reduction in the amplitude of the swings. In many cases the phenomena can be quantitatively explained, with an approximation quite good enough for every practical purpose, if one assumes that the resisting couple has a moment equal at every instant to the product of a constant of the apparatus and the angular velocity which the body then has; and more than seventy years ago Gauss and W. Weber gave an exhaustive mathematical treatment,[2] based upon this hypothesis, of the behavior of such swinging magnets as they employed in their magnetic measurements at Göttingen. It appeared from their analysis, which in simplified form is given in most modern treatises on Physics, that if the resistance follows the law stated above, the ratio of any two successive elongations of the magnet must have a constant value; and they used the natural logarithm (λ) of this ratio, under the name of the "logarithmic decrement" of the motion, in many of their equations.

The resistance which air, under given conditions of temperature, pressure, and confinement, offers to a body of given form and dimensions, moving through it at a *uniform* velocity, v, has been studied by

[1] *Proceedings of the American Academy of Arts and Sciences*, November, 1908; vol. xliv, no. 2.

[2] Gauss, *Resultate des Magnetischen Vereins*, 1837; W. Weber, *Resultate des Magnetischen Vereins*, 1837, 1838; *Maassbestimmungen*, vol. ii; *Math.-phys. Abhandlungen der K. Sächs. Gesellschaft*, 1852; Du Bois-Reymond, *Monatsberichte der Berl. Akad.*, 1869, 1870.

a great number of experimenters under a great variety of physical conditions, and a résumé of the results at which they have arrived can be found in the articles of Finsterwalder on Aërodynamik and of Cranz on Ballistik in the fourth volume of the Encyklopädie der Mathematischen Wissenschaften.[1]

That under otherwise given conditions the air resistance, when v is large, is a complicated function of v, is shown by the practical formulas based on experiments made with rotating projectiles of the standard Krupp form. For a projectile of this kind of given size, in free air, the expressions are av^2, bv^3, cv^5, dv^3, ev^2, $fv^{1.7}$, $gv^{1.55}$, according as v, measured in meters per second, lies in one or other of the intervals between the values 50, 240, 295, 375, 419, 550, 800, and 1000. The constants are different for projectiles of different diameters and vary with the temperature of the air, the barometric pressure, and other circumstances.

In order to determine the resistance which the air offers to a given body moving uniformly through it at a comparatively small velocity, v, many different observers have made use of the whirling table in some form. The phenomenon to be studied is in any case a very complex one, since the moving body drags with it, as it moves, a certain mass of air, and the viscosity of the air contributes an uncertain amount to the quantity to be measured. It appears, however, from the experiments of Schellbach, von Loessl, Langley, Recknagel, Hagen, and others,[2] that when proper corrections have been made for the effect of the wind which the table takes with it as it turns, the air resistance varies as the square of the velocity [3] for all values of v between 50 and 0.2. For velocities much less than 20 centimeters per second the viscosity of the air appears to determine the resistance which is

[1] Leipzig, B. G. Teubner, 1903.

[2] Schellbach, *Ann. d. Phys.*, vol. cxliii, 1871; Recknagel, *Zeitschrift d. Ver. deutsch. Ing.*, vol. xxx, 1886; F. v. Loessl, *Die Luftwiderstandsgesetze;* Langley, *Experiments in Aërodynamics;* Cranz, *Aeussere Ballistik*, Leipzig, 1895; Thiesen, *Ann. d. Phys.*, vol. xxvi, 1885; Mach und Salcher, *Wiener Berichte*, 1887, 1889.

[3] Mohn, *Grundzüge der Meteorologie*, Zweite Auflage, p. 137: "Durch vergleichende Versuche über Druck und Geschwindigkeit des Windes, hat man gefunden dass der Winddruck dem Quadrate der Geschwindigkeit proportional ist." On page 138, however, the pressure of the wind in kilograms per square meter is given as 0.15, 1.87, 5.96, 15.27, 34.35, 95.4, according as the velocity in meters per second is 0.5, 4, 7, 11, 17, or 28.

approximately proportional to the velocity. It is well to remember that a solid sphere, to take a concrete example, moving in an infinite homogeneous liquid at rest at infinity, in a straight line, with *constant* velocity, would encounter no resistance from the liquid if there were no viscosity; but that even in a homogeneous, perfect liquid, a sphere moving with *changing* velocity would meet with a resistance from the liquid, and the inertia of the sphere would in consequence of this be apparently increased in a manner which could be mathematically accounted for in the equation of motion of the sphere, if the mass of the sphere were increased by half the mass of the displaced liquid.

If at the point (x, y, z) in a viscous fluid at the time t the components of the velocity are u, v, w, if the applied body forces which urge the fluid have the components X, Y, Z, if ρ is the density, and if μ represents a constant of the fluid which measures its coefficient of viscosity, the equations of motion of the fluid as established by Navier and Poisson [1] are usually written in the forms:

$$\rho\left(\frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial x} + v\cdot\frac{\partial u}{\partial y} + w\cdot\frac{\partial u}{\partial z}\right) = \rho X - \frac{\partial p}{\partial x} + \tfrac{1}{3}\mu\cdot\frac{\partial \bar{\omega}}{\partial x} + \mu\cdot\nabla^2(u),$$

$$\rho\left(\frac{\partial v}{\partial t} + u\cdot\frac{\partial v}{\partial x} + v\cdot\frac{\partial v}{\partial y} + w\cdot\frac{\partial v}{\partial z}\right) = \rho Y - \frac{\partial p}{\partial y} + \tfrac{1}{3}\mu\cdot\frac{\partial \bar{\omega}}{\partial y} + \mu\cdot\nabla^2(v),$$

$$\rho\left(\frac{\partial w}{\partial t} + u\cdot\frac{\partial w}{\partial x} + v\cdot\frac{\partial w}{\partial y} + w\cdot\frac{\partial w}{\partial z}\right) = \rho Z - \frac{\partial p}{\partial z} + \tfrac{1}{3}\mu\cdot\frac{\partial \bar{\omega}}{\partial z} + \mu\cdot\nabla^2(w), \tag{1}$$

where $$\bar{\omega} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

and p represents the arithmetical mean of the normal pressures on any three mutually perpendicular planes through the point (x, y, z).

Using these equations, Stokes, in a paper [2] presented to the Cambridge Philosophical Society in December, 1850, determined the resistance which a sphere making small harmonic oscillations of complete period T, in an infinite viscous liquid, would encounter, and showed that if θ represented the distance of the centre of the sphere

[1] Navier, *Mémoire de l'Académie des Sciences*, vol. vi, 1822; Poisson, *Journal de l'École Polytechnique*, vol. xiii, 1829.

[2] Stokes, Mathematical and Physical Papers, vol. ii.

from its mean position at the time t, the value of this resistance would be

$$\left(\frac{1}{2}+\frac{9}{4af}\right)M'\cdot\frac{d^2\theta}{dt^2}+\frac{9\pi M'}{2afT}\left(1+\frac{1}{af}\right)\cdot\frac{d\theta}{dt}, \tag{2}$$

or

$$M''\frac{d^2\theta}{dt^2}+2m\frac{d\theta}{dt}, \tag{3}$$

where a is the radius of the sphere, M', the mass of the displaced liquid, and $f^2=\pi\rho/\mu T$: M_0 is the mass of the sphere.

Such a sphere, oscillating under the action of this resistance and a restoring force ($b^2\theta$) proportional to the displacement, would have an equation of motion of the form

$$M\cdot\frac{d^2\theta}{dt^2}+2m\cdot\frac{d\theta}{dt}+b^2\theta=0, \tag{4}$$

where $M=M_0+M''$: all the coefficients are to be considered constant, since b^2 is fixed, but they would be different for a different period of oscillation.

For an infinitely long cylinder of revolution also, oscillating in a viscous liquid, in a direction perpendicular to the axis of the cylinder, Stokes found an equation of motion of this same familiar form which had long been used to explain the behavior of pendulums, though it had been founded on a theory quite different from his. As early as 1828 Bessel [1] had pointed out the necessity of allowing for the inertia of the air which accompanies a pendulum in its motion, and the work of Sabine, Dubuat, Poisson, Baily, Plana, South, and others, had made it clear that in practical cases the moment of inertia of the swinging system might be twice that of the pendulum bob, and that the "resistance" of the air might be accounted for in many practical cases by assuming it to be proportional to the first power of the angular velocity. This equation had been used by Gauss for determining the motion of swinging bar magnets, as has been already mentioned, and it still forms the foundation of much modern work, as, for in-

[1] Bessel, *Untersuchungen über die Länge des einfachen Secunden Pendels*, Berlin, 1828; Bottomley, *Phil. Mag.*, vol. xxiii, 1887; Graetz, Reibung, Winkelmann's Handbuch der Physik, i; O. E. Meyer, *Pogg. Ann.*, vol. cxiii, 1861; vol. cxxv, 1863; vols. cxlii, cxliii, 1871; vol. cxlviii, 1873; *Wied. Ann.*, vol. xxiii, 1887; Kundt and Warburg, *Pogg. Ann.*, vol. clv, 1875; Crookes and Stokes, *Proceedings Royal Society*, 1888.

stance, that on the properties of damped d'Arsonval galvanometers.[1]

If, however, a swinging magnet presents to the air a relatively larger surface, or if the magnet is provided with a large mica damping vane, it often happens that the resistance of the air cannot be satisfactorily explained on the assumption that it is proportional to the angular velocity at every instant, and that at the beginning of the motion it seems to be much more nearly proportional to the square of the angular velocity. It will be convenient, therefore, to consider first the manner in which the amplitude of an oscillating body would decrease if the motion were resisted by a couple of moment proportional to the square of the angular velocity. A roughly approximate solution of this problem was printed by Poisson in 1811, but is not accurate enough for practical purposes. We shall do well to attack it in another way.

If θ is the angular deviation in radians of the moving body from the position of equilibrium, and $b^2\theta$ the restoring moment, the moment of the couple due to the resistance of the air is of the form $2a(d\theta/dt)^2$; and if K represents the moment of inertia of the swinging system, the equation of motion is

$$K \cdot \frac{d^2\theta}{dt^2} + 2a\left(\frac{d\theta}{dt}\right)^2 + b^2\theta = 0, \tag{5}$$

or

$$\frac{d^2\theta}{dt^2} + 2a\left(\frac{d\theta}{dt}\right)^2 + \beta^2\theta = 0, \tag{6}$$

when the body is swinging in the positive direction.

If for $d\theta/dt$ we write ω, $d^2\theta/dt^2$ is equal to $\omega . d\omega/d\theta$, and the equation becomes

$$\omega \cdot d\omega + (2a\omega^2 + \beta^2\theta)\, d\theta = 0, \tag{7}$$

which will become exact if we multiply through by $e^{4a\theta}$, so that,

$$\omega^2 = 2c \cdot e^{-4a\theta} + \frac{\beta^2}{8a^2} - \frac{\beta^2\theta}{2a}, \tag{8}$$

or

$$\omega^2 = 2ce^{-k\theta} + m - mk\theta, \tag{9}$$

where c is a constant of integration.

[1] Dorn, *Ann. der Physik*, vol. xvii, 1882; vol. xxxv, 1888. F. Kohlrausch: "Ueber die Inconstanz der Dämpfungsfunction eines Galvanometers und ihren Einfluss auf die Absolute Widerstandsbestimmung mit dem Erdinductor," *Ann. der Phys.*, vol. xxvi, 1885; Schering, *Ann. der Phys.*, vol. ix, 1880; Jaeger, *Instrumentenkunde*, 1903; Dorn, *Ann. der Physik*, vol. xvii, 1882.

If $-\theta_0$ is the value of the angular deviation at any elongation on the negative side, and if θ_1 is the next elongation on the positive side, then, for the same value of c,

$$2ce^{k\theta_0} + m + mk\theta_0 = 0, \tag{10}$$

$$2ce^{-k\theta_1} + m - mk\theta_1 = 0, \tag{11}$$

or

$$(1 + k\theta_0)\, e^{-k\theta_0} = (1 - k\theta_1)e^{+k\theta_1}, \tag{12}$$

where $k = 4a$. This equation does not involve β.

For swings of large amplitude, it is easy to find θ_1 graphically, when k and θ_0 are given, by aid of this last equation. When θ_0 is small, however, we may, in any practical case, develop each number of (12) in a very convergent power series of which we need keep only terms of order lower than the fourth.

This procedure gives the equation

$$2k(\theta^3{}_0 + \theta^3{}_1) - 3(\theta^2{}_0 - \theta^2{}_1) = 0, \tag{13}$$

which is satisfied when $\theta = -\theta_0$ and from this we may find, by aid of a second development, the very approximate result,

$$\theta_1 = \theta_0 - \tfrac{2}{3}k\theta_0{}^2. \tag{14}$$

If terms of the fourth order are kept, we may obtain the expressions

$$\theta_1 = \theta_0 - \tfrac{2}{3}k\theta^2{}_0 + \tfrac{4}{9}k^2\theta^3{}_0, \tag{15}$$

$$\theta_2 = \theta_0 - \tfrac{4}{3}k\theta^2{}_0 + \tfrac{16}{9}k^2\theta^3{}_0,$$

but for most practical purposes (14) is quite accurate enough.

After the swinging system has come momentarily to rest at the elongation $-\theta_0$, it moves in the positive direction with an angular velocity which increases to a maximum at a position determined by the constants of the motion, and has the value ω_0 when $\theta = 0$.

It is easy to see from (3) that

$$\omega_0{}^2 = 2c + m, \tag{16}$$

and from (9) that

$$2c = -m(1 + k\theta_0)\, e^{-k\theta_0}, \tag{17}$$

so that

$$\omega_0{}^2 = m - m\,(1 + k\theta_0)e^{-k\theta_0}; \tag{18}$$

and it is evident that ω_0 is greater, other things being given, the greater the amplitude of the motion; that is, the greater the value of θ_0. Equation (16) shows, however, that the greatest value which ω_0 can

have is $\sqrt{m}$, and it is interesting to determine what elongation on the positive side of the zero point corresponds to this angular velocity at $\theta = 0$.

If in (12) we suppose θ_0 to grow large without limit, θ_1 approaches the limit $1/k$, and it appears that however great the angle through which the swinging system may have been turned out of the position of equilibrium at the outset, the amplitude of the next elongation cannot be greater than $1/k$th of a radian, and the next turning point to this (on the same side of the zero as the original disturbance) must come at an angular distance from the position of equilibrium not greater than about $0.594/k$ radians. The subsequent swings decrease regularly in amplitude in such a manner as to make the logarithmic decrement decrease towards zero. At any time during the motion the determination of two successive amplitudes serves to determine k through (12), for it is easy to solve the transcendental equation to any desired accuracy.

If we differentiate (6) with respect to t, and represent $d\omega/dt$ by r, we shall get the equation

$$\frac{rdr}{4\alpha r + \beta^2} + \omega d\omega = 0, \tag{19}$$

or

$$r - \frac{\beta^2}{4\alpha} \log\left(r + \frac{\beta^2}{4\alpha}\right) = C - 2\alpha\omega^2, \tag{20}$$

and C, the constant of integration, may be determined from a consideration of the fact that when $\omega = 0$, θ is $-\theta_0$.

If a swinging system oscillates about a position of equilibrium under the action of a righting moment proportional to the deviation and a resisting couple proportional during the whole motion to the *first* power of the instantaneous angular velocity, the equation of motion has the familiar form

$$\frac{d^2\theta}{dt^2} + 2\alpha \frac{d\theta}{dt} + \beta^2\theta = 0. \tag{21}$$

If $\rho^2 = \beta^2 - \alpha^2$, and if m and n are the roots of the equation

$$x^2 + 2\alpha x + \beta^2 = 0, \tag{22}$$

$$m = -\alpha + \rho i, \qquad n = -\alpha - \rho i,$$

and we have

$$\theta = e^{-\alpha t}(L \cos \rho t + M \sin \rho t), \tag{23}$$

or $$\theta = Ae^{-\alpha t} \sin (\rho t - e), \tag{24}$$

where A and e are constants of integration. If, using t and θ as coordinates, we plot (24), it is clear that the curve $\theta = Ae^{-\alpha t}$ touches the curve $\theta = Ae^{-\alpha t} \sin (\rho t - e)$ when $\rho t - e = (2k + \frac{1}{2})\pi$, so that if the time be counted from the date of one of these points of tangency, the corresponding solution of (21) may be written in the form

$$\theta = Be^{-\alpha t} \cos \rho t \tag{25}$$

The complete period of the oscillation (T) is $2\pi/\rho$. The ratio of the amplitudes at two consecutive elongations is $e^{\alpha\pi/\rho}$ and the logarithmic decrement is $\alpha\pi/\rho$. The ratio of the amplitudes at two consecutive elongations on the same side of the position of equilibrium is $e^{2\alpha\pi/\rho}$, and we have

$$\alpha = 2\lambda/T, \qquad \beta^2 = 4(\pi^2 + \lambda^2)/T^2. \tag{26}$$

The maxima of the curve (24) occur at times defined by the equation

$$\tan (\rho t - e) = \rho/\alpha; \text{ or } \sin (\rho t - e) = \rho/\beta,$$

and the curve $$\theta = \frac{A\rho}{\beta} e^{-\alpha t}$$

passes through all these points, which are spaced at equal time intervals T.

If, then, the curve which represents as a function of the time (t) the deviation (θ) of a swinging body from the position of equilibrium be drawn, and if the motion be of the kind defined by the equation (21), the maxima will be spaced at equal time intervals, and it will be possible to pass through all the crests a curve of the family $\theta = Ce^{-\alpha t}$ where C and α are constants. It is easy to see whether or not this last condition is satisfied in any given case, if one has measured a series of successive amplitudes on the same side ($d_1, d_2, d_3, d_4, d_5, \ldots d_r$). If we measure t from the date of the first of these elongations, the desired curve must have an equation of the form $\theta = d_1 e^{-\alpha t}$, and αT may be determined from any other amplitude (say the kth) for

$$d_k = d_1 e^{-(k-1)\alpha T}.$$

If the value of αT thus found be the same for all values of k, the condition is satisfied. Sometimes when the period of the oscillation is extremely short, the maximum points seem to form a continuous curve, unless the diagram be much drawn out horizontally. In such a

case as this one may use, in making the test just described, not a series of successive amplitudes on the same side of the position of equilibrium, but points on the curve, taken at convenient values of t equally spaced.

The Damping of the Quick Oscillations of a Light System suspended between two Stretched Wires by the Resistance of the Air and Frictional Forces in the Wire

It will appear from the observations recorded in this paper that if a small magnetic needle be mounted horizontally with a minute galvanometer mirror upon a short, stiff, vertical piece of wire or glass filament stretched between two vertical pieces of fine wire, and if the needle be turned horizontally out of its position of rest through an angle of say 5° and then allowed to oscillate, the curve drawn through the crests of the oscillations as represented on a photograph record will usually not coincide exactly with any exponential curve of the family mentioned above. If a curve of this family be drawn nearly through a number of crests in the middle of the diagram, it will usually fall somewhat below the observed curve at each end. It will be convenient to instance a few typical cases at the outset.

I. Figure 1 (Plate 1) is a copy of a photographic record obtained from a short-period mirror galvanometer. The one-centimeter-long needle of this instrument, made of watch spring, was mounted on a short, stout, inflexible piece of glass fibre, together with a minute bit of very thin mirror, and the fibre was suspended, like the coil of a marine d'Arsonval galvanometer,[1] between two pieces of extremely fine gimp, under gentle tension. The light from an electric projecting lantern about twenty feet from the galvanometer, shining through a small hole in a brass plate used as a lantern slide, fell upon the galvanometer mirror, and a sharp image of the hole was formed on a piece of sensitive paper on a horizontal revolving drum at a considerable distance from the mirror. The needle was first deflected a little off scale by a steady current sent through the coils of the galvanometer when the light was screened, the screen was then removed and a record of the manner of decay of the amplitude of the excursions of

[1] See *M*, Figure 3.

the needle obtained when the galvanometer circuit had been suddenly broken. The moment of the couple due to the mutual action of the magnet and the earth's field was relatively inappreciable. Three different drums and three pieces of chronograph clock work were used in making the records discussed in this paper, but for the fastest speeds an electric motor driving a worm gear accurately cut for the purpose by Mr. G. W. Thompson, the mechanician of the Jefferson Laboratory, was employed, and this left nothing to be desired. The apparatus was put together by Mr. John Coulson, who helped me in all the work and took many of the photographic records. Most of these records, of which I have a very large number, were about 50 cms. long and 20 cms. wide, but much larger ones could be obtained if desirable.

On one of the photographs taken with the apparatus just described a series of measurements of the amplitudes of the oscillations, as depicted on the diagram, were made at times, represented by whole centimeters from the time origin, on the figure. The successive values for the excursions were: 1260, 1006, 791, 646, 521, 420, 349, 280, 231, 190, 159, 131, on a scale of equal parts, and, at the scale distance used, these numbers were accurately proportional to the angular amplitudes of the needle at the times concerned. If, then, the resistance to the oscillations were proportional to the angular velocity, it should be possible to draw a curve of the family $y = A \cdot e^{-\alpha t}$ the successive ordinates of which, taken at the proper time interval (T), should have the lengths indicated above.

If, however, we assume that when $t = 0$, $y = 1260$, and use the other numbers given above, in succession for determining αT, we get for this product the different values 0.225, 0.238, 0.224, 0.221, 0.220, 0.217, 0.214, 0.212, 0.210, 0.207, and it is evidently impossible to find the curve sought exactly. The differences, while much too great to be accidental, are intrinsically not very large, and a curve of the family $y = A \cdot e^{-\alpha t}$ may be drawn which will pass through the fourth, fifth, and sixth points, and the ordinates of which at the ends of the series will be 1231 and 116, instead of 1260 and 131. Corresponding ordinates of the observed and calculated curves are shown in the following table. The calculated curve has, as a whole, a less curvature than the observed curve, and the ratio of any excursion to the next decreases slightly with the time. The period was about 1/100th of a second.

In the case of this particular quickly oscillating system, therefore, the first double amplitude of which was not larger than 6°, the motion can be explained during a considerable part of its course with fair approximation on the assumption that the resistance due to the air and to frictional forces in the fibre is proportional to the angular

TABLE I

Observed	Calculated	Observed	Calculated
1260	1231	349	339
1006	993	280	273
791	801	231	221
646	646	190	178
521	521	159	143
420	420	131	116

velocity. The deviations from this law, while real, are not greater than one often finds in the motion of a suspended magnet or the coil of a d'Arsonval galvanometer, when swinging slowly over a small arc. Indeed two d'Arsonval galvanometers of the same type and apparently very like each other may depart from the usual law in opposite directions if the periods are long; in one the logarithmic decrement may grow larger as the amplitudes of a long series of swings decay, while in the other it may become smaller. In the case of a galvanometer of this kind in the Jefferson Laboratory, the ratio of one excursion to the next *increased* from 1.063 to 1.086 in an hour and a quarter, while the amplitude decreased to about four tenths of its original value. The complete period of swing was 158 seconds. A similar galvanometer in the same room has a coil the swings of which decay at a *decreasing* rate as the amplitudes grow less.

In his *Anleitung zur Bestimmung der Schwingungsdauer einer Magnetnadel* (1837), Gauss describes a suspended magnet the logarithmic decrement of the swings of which increased on a certain occasion from 1168×10^{-6} to 1301×10^{-6} in 422 oscillations. The actual value of the logarithmic decrement for this magnet and for a given amplitude varied from day to day, being usually smaller in cloudy weather.

II. After a number of records had been made like that reproduced in Figure 1, a small vertical mica damping vane of about 3 square centimeters area, was fastened symmetrically to the little glass rod which carried the mirror of the swinging system, and a new series of

records were obtained. The restoring moment was the same as before, but the moment of inertia had been increased somewhat, as well as the resistance due to the air. Under these circumstances the period was much longer than before, while the manner of decay of the amplitudes was much the same. Figure 2 (Plate 1) represents on a reduced scale one of the smaller photographs. Figure A was plotted from a large

FIGURE A

record in which the crests of successive oscillations were 4.5 millimeters apart at the beginning of the diagram and nearly 4.9 millimeters at the end. Such a gradual change of period during the motion often accompanies the swinging of a magnet under the torsional forces of a stretched wire.[1]

The values, in ten thousandths of a radian, of a number of successive amplitudes, as obtained from the photograph, were: 597, 556,

[1] Guthe, *Physical Review*, 1908.

518, 481, 448, 419, 390, 367, 341, 320, 300, 280, 262, 246, 230, 217, 203, 190, 179, 168, 159, 148, 140, 132, 124, 117, 111, 104, 98, 92.

These numbers, used as ordinates of points with equally spaced abscissas, give a curve of the form shown in Figure A by the full line WHCDK. The dotted line VCDG shows a curve of the family $y = A \cdot e^{-\alpha t}$, which coincides almost exactly with the full line between the points C and D.

The curves HT, CL represent attempts to determine the constants of an equation of the form (6) which should yield a curve of amplitudes like the observed curve. Both HT and CL pass exactly through two adjacent points of the line WHCDK, and the other points were determined by a series of applications of the equation (8). Some of the characteristics of certain of the records which I obtained resemble those of oscillations under a resistance proportional to the square of the angular velocity, but it is evident that in the case here considered the resistance does not nearly follow this law. We may notice that according to Poisson's rather rough approximation, HT and CL would be straight lines.

III. A new suspended system was then made of two 15-millimeter-long magnetic needles mounted horizontally, one over the other (together with a small mirror), on a short bit of glass fibre stretched

TABLE II

1705	1509	418	411
1445	1341	379	365
1230	1141	342	324
1075	1058	313	288
940	940	282	256
828	835	255	227
734	742	230	202
652	658	208	179
582	586	185	159
520	520	174	142
465	462		

between two short lengths of No. 36 steel wire. The restoring forces came from the torsional forces in these wires. The mirror and needles together exposed to the air a resisting surface of less than a square centimeter area. The period was about 1/48th of a second. The numbers in the first column of the next table show the lengths of successive ordinates (taken at equal time intervals) of the curve drawn

on the photographic record through the crests of the oscillations. The next column gives the lengths of the corresponding ordinates of a curve of the family $A \cdot e^{-at}$ drawn exactly through the fifth and tenth crests. The very tips of the needles at the beginning of the motion passed over about 10 centimeters of path per second.

IV. Figure B shows the manner of decay of the oscillations of a light suspended system under the action of very strong restoring forces. A small mirror and two 15-millimeter long watch-spring magnets were mounted on a square vertical mica vane, of about 3 square

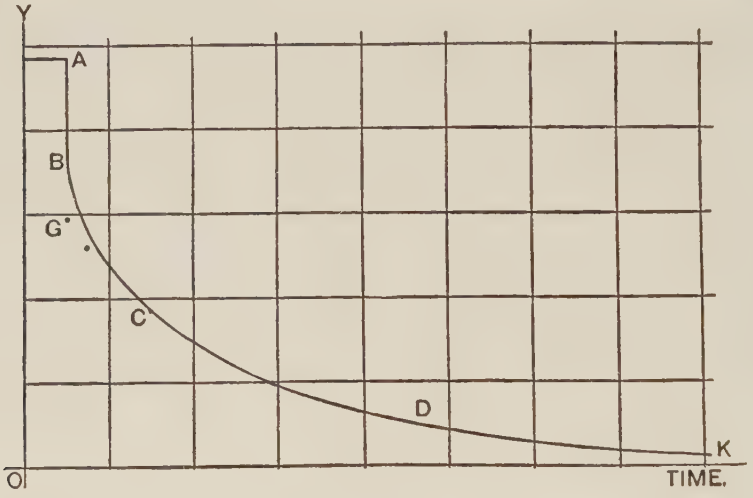


FIGURE B

centimeters area, which was fastened symmetrically on a slender but stiff bit of glass filament. The filament was stretched between two pieces of No. 36 B. & S. steel wire about 2 centimeters long. The righting moment was due partly to the torsional forces in the wire and partly to a strong electromagnetic field about the needles. When the circuit of the magnetic field used to deflect the needle through the initial angle θ_0 was suddenly broken, the vane and its belongings moved quickly (in perhaps 1/250th of a second) through the position of equilibrium and out on the other side to a turning point corresponding to a deviation of about three fourths of θ_0. After this the amplitudes decreased slowly and continuously. The curve drawn through the crests of the oscillations consists at the start of a vertical line, as

it would if, for instance, the resistance followed the law of the square of the angular velocity. After a short time, however, the curve, like most of those which I have obtained, follows more nearly a course which corresponds to the equation $y = A \cdot e^{-at}$. The numbers in the next table show well enough what the character of the agreement is. The first column gives ordinates of the photographic record taken at equal time intervals. The second column gives corresponding ordinates of a curve of the family $y = A \cdot e^{-at}$ which falls in very nearly with the first curve for a portion of the middle of its course.

TABLE III

4840–3750	2950	1100	1095
2950	2695	1012	1001
2560	2463	930	915
2295	2251	862	836
2065	2057	800	764
1880	1880	745	698
1710	1718	695	639
1560	1570	645	583
1430	1435	600	533
1315	1312	...	...
1200	1200	390	310

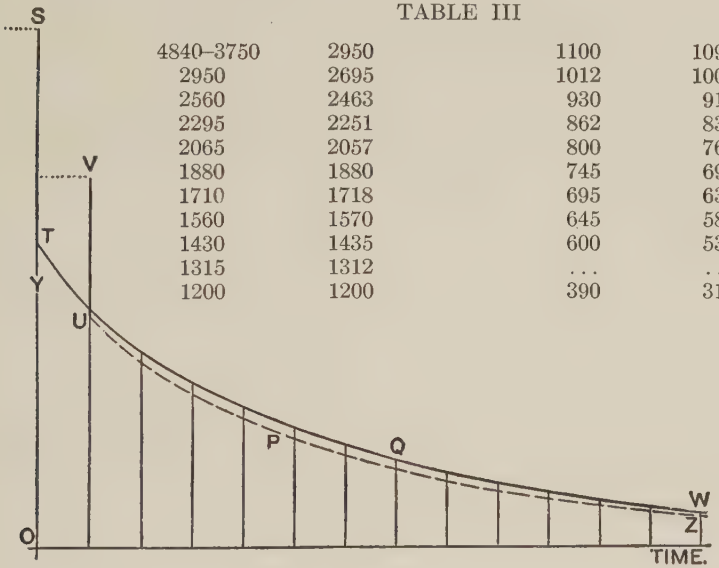


FIGURE C

V. Figure C represents curves taken with this apparatus when the filament was made of a piece of manganine wire. One curve is here displaced an arbitrary amount with respect to the other, for purposes of comparison. The sudden drop (ST) from the original deflected position to one of much smaller displacement, after which the decrease of amplitude is gradual, is clearly shown. The two curves show different values of the original deflection.

The Damping of the Slow Oscillations of a d'Arsonval Galvanometer Coil, which is Wound on a Nonmetallic Core, and is swinging between the Poles of its Magnet

If the coil of a d'Arsonval galvanometer be wound on a wooden spool, and if its circuit be open, the damping of its oscillations is due principally, unless the copper wire is magnetic, to air resistance, and only slightly to frictional forces within or at the surface of the gimp from which the coil hangs. When, however, the circuit of the coil is closed through an outside resistance x, electromagnetic damping is added, and the damping coefficient of the motion is larger than before, or, if x is small enough, the motion ceases to be periodic. In many instances it is possible and desirable to damp the coil critically, but this is sometimes impracticable, — as, for instance, in such instruments of long period (400 or 500 seconds) as are used in testing massive iron cores — and there are certain kinds of absolute measurements where a relatively undamped instrument is preferable. The throw of a d'Arsonval galvanometer due to a given change of the flux of magnetic induction through its circuit is usually to be quantitatively explained only by attributing to the resistance of the circuit a value much greater than the real one. This apparent resistance [1] may be many times as great as the real resistance; its value depends upon the constants of the motion of the coil, and it not infrequently happens that a knowledge of these "constants," is important, even though the amplitudes do not always decrease exactly according to the assumption that the resistance to the motion is equivalent to a couple of moment proportional to the angular velocity.

If a coil of the ordinary Ayrton-Mather form, without a damping vane, swing between the poles of its magnet with the coil circuit open, the amplitude generally decreases slowly, and if the coil be hung successively by pieces of gimp of different lengths or stiffnesses, the period changes with the restoring moment, and the damping coefficient (α) remains small, though it often changes somewhat with the amplitude. If with a given suspension we determine the quantity α in the equation $y = A \cdot e^{-\alpha t}$ from two amplitudes of about 5° near the be-

[1] Robinson, *The Electrician*, 1901; White, *Physical Review*, 1904; Peirce, *Proceedings of the American Academy*, 1906.

ginning of the motion, and then from two amplitudes of about 2° after the coil has made twenty or thirty swings, the latter value will usually be sensibly smaller than the other, but the difference is not very great unless the restoring force is weak, as it is in very sensitive instruments.

VI. In the case of a certain galvanometer of the Ayrton-Mather type which I studied at length, the value of α fell from 0.00403 to 0.00356 as the motion progressed, when a piece of very fine steel gimp was used to hang the coil. When stiff gimp was employed, the value of α remained much more nearly constant while the amplitude decreased, and was nearly the same for different lengths of the gimp. The first column in the next table shows the period as determined principally by the stiffness of the gimp, the second column gives the corresponding value of α determined after twenty or thirty swings had been executed and the double amplitude had fallen below 4°.

TABLE IV

T	Damping Coefficient
2.59	0.0029
3.62	0.0028
4.57	0.0031

The resistance of the instrument was about 21 ohms, but a considerable fraction of this was in the gimp.

When the coil circuit was closed by a resistance of 400 ohms, and the coil was hung successively by several different pieces of gimp of different lengths, the damping coefficient (α) slowly decreased as the amplitude decreased, so that the logarithmic decrement was not quite constant during the whole motion in any case, but the value of α for a double amplitude of say 4° was practically the same for widely different periods.

The next two tables show the results of measurements of a good number of photographic records. In the first case, as has been said, the outside resistance of the circuit was 400 ohms, in the second case it was 200 ohms.

If the coil was deflected out of its position of equilibrium through an angle of perhaps 10°, and was then suddenly released, the amplitude fell at once to a much smaller value, especially when the coil was closed through a resistance of say 400, and then decreased gradually

in much the same manner as the swings represented by Figure 5. The phenomenon is, however, not so marked as when the damping is fairly large and due wholly to air resistance.

When the circuit of the coil of a d'Arsonval galvanometer of the form described is closed through an outside resistance, x, so that the whole resistance is $(g + x)$, the damping coefficient of the motion is theoretically the sum of the corresponding coefficient when the circuit is open and the coefficient which the electromagnetic damping would

TABLE V

T	Damping Coefficient
9.28	0.0113
4.57	0.0113
3.62	0.0113
2.61	0.0114

TABLE VI

T	Damping Coefficient
7.58	0.0193
4.57	0.0192
3.62	0.0192
2.61	0.0191

cause if the air damping were absent, and this last should be proportional reciprocally to the apparent resistance $(g' + x)$ of the circuit where g' is usually considerably larger than g. A set of five photographic records were obtained with the coil mentioned above when it was suspended by a certain short piece of wire which gave the system a period of about 2.60 seconds. The next table shows (1) the values of x, (2) the corresponding values of α determined by a series of measurements of the diagrams from amplitudes not greater than 4°, (3) the values which the damping coefficient (α') would have if the air damping were absent, as calculated by aid of Table IV, and finally the reciprocal (y) of α'. Since α' should theoretically be of the form

$$\frac{.1}{y} = \frac{b}{x + g'}, \tag{29}$$

if the observed values of x and y be plotted, the locus should be a straight line the intercept of which on the axis of abscissas is the value of g'.

TABLE VII

x	α	α'	y
400	0.0114	0.0085	11.76
200	0.0191	0.0162	6.17
100	0.0358	0.0315	3.24
50	0.0595	0.0552	1.81
20	0.1120	0.1091	0.92

As a matter of fact, the points indicated by this table lie almost exactly on a line which cuts the x axis at a point the abscissa of which is a little less than forty. The apparent resistance of the galvanometer is, therefore, a trifle less than 40 ohms, while its real resistance with this wire is less than 20 ohms.

The Motion of a Suspended System which carries a Relatively Large Damping Vane under Righting Couples of Different Strengths

In order to study the effects of different restoring moments upon a swinging system furnished with a given damping vane, I used the apparatus represented in Figure 3 (Plate 2). G is a uniformly wound solenoid the horizontal axis of which lies in the meridian at a place where H is known. From a fine fibre in a narrow chimney inserted in the top of the solenoid at the centre hangs a small bar magnet (Q) fastened to a stiff mica vane in the manner shown at N. The axis of the magnet is coincident with the axis of the solenoid. A small mirror on the vertical wire which carries the vane and magnet lies in a vertical plane which makes an angle of 45° with the vertical plane through the axis of the solenoid, and, receiving the light from a small round hole in a brass plate in the slide holder of a distant Schuckert projecting lantern, throws it upon a sheet of bromide paper wound upon the drum D, where a small very sharp image of the hole is formed. The drum may be turned uniformly at very various speeds, either by clockwork or by an alternating motor actuated by a 60 cycle, 110 volt street circuit. The magnetic field about the suspended magnet can be given any desired value within wide limits by sending through G a suitable steady current from a battery of large storage cells. A current from another similar battery sent through the coil K serves to deflect the magnet out of the meridian against the given

restoring field. When the current in K is suddenly interrupted, the suspended system oscillates with continually decreasing amplitude about the horizontal meridian and makes a record of its motion upon the photographic paper. In order that the seam in the paper on the drum may not come at an undesirable place in the record, the break in K's circuit is made automatically by the drum when it reaches a given position, but the system of relays by which this is accomplished is not indicated in the figure.

Experience gained with this apparatus shows that if the original deflection caused by a steady current in K is not more than 5° or 6°, and if the intensity of the magnetic field about the magnet is not too great, the record obtained after K's circuit has been suddenly broken is such that it is possible to draw a curve of the family $y = A \cdot e^{at}$ which shall, within the errors of observation, pass through all the crests of the diagram except the first two or three. We may assume that the motion in a case like this could be mathematically explained on the assumption that a body of fixed moment of inertia (I), — quite different, however, from the moment of inertia of the actual suspended system swinging in vacuo, — is oscillating under the action of the restoring moment due to the magnetic field and a retarding moment equal at every instant to the product of a damping coefficient ($2a$) and the angular velocity of the system. If the intensity of the field about the magnet be somewhat changed, I will have nearly its old value, but the damping coefficient, though constant for a given system swinging with a given period, has a new value when the period is changed. The change of the damping coefficient usually follows the direction of the change indicated by Stokes's theoretical treatment of the resistance encountered by a sphere making harmonic oscillations of small amplitude in a viscous liquid. It is usually rather difficult to determine the apparent moment of inertia of the system (I) with accuracy from observations of the period of the oscillations (for there generally is a fixed period), the value of the damping factor, the intensity of the external magnetic field about the magnet, and the moment of the magnet in that field, but such values of I as my observations give do not seem to change in any such manner as Stokes's formula for the sphere demands. Of course the two cases are mathematically quite different.

If the magnetic field about the magnet is relatively intense, and if the original deflection is as great as 10°, the system swings through its position of equilibrium, when it is released, to an elongation on the other side only a fraction (perhaps a half or a quarter) of the original deflection. From this time on the amplitude decreases slowly and regularly, much as in the case figured in Diagram C.

If a seasoned magnet placed in G be subjected to a magnetic field of several units' strength, the magnetic moment changes, and it is necessary to determine the amount of this change with some care if one needs to know the restoring couple which acts upon the swinging system. I have used for measurements of this kind a simple induction-coefficient apparatus shown diagrammatically in Figure 4 (Plate 2). P and Q are two similar solenoids which may be set anywhere on a horizontal east-west track *vw*. O is a mirror magnetometer the deflections of the needle of which can be determined by the telescope and scale (T, S). A horizontal scale *ab* in the meridian carries a wooden holder which contains a seasoned magnet (M_0) protected from sudden temperature changes, in Gauss's B Position with respect to the magnetometer needle. P and Q are so connected in series with a storage battery, a rheostat, and a standard centiamperemeter that a current can be sent in opposite directions through the solenoids; it is then easy, when a current stronger than any to be used in the subsequent determinations is passing through the circuit, to arrange the positions of P and Q near O on *vw*, so that the current shall not affect the needle. After this adjustment has been made, the magnet to be tested is placed in P somewhere near the middle of the solenoid and so near the needle that the latter is deflected off scale, and the wooden holder containing M_0 is placed on *ab* at such a distance from the needle that the latter is brought back exactly to its undeflected position. If then a current of suitable, small intensity be sent through the solenoid circuit, the change of the moment of the magnet in P from M to M' causes a scale reading z owing to a deflection (δ) of the needle; and if the current has not been too strong, this deflection disappears when the circuit is broken. If the field to which the magnet has been exposed has been fairly large, however, the moment is permanently changed by a small amount, and it is then necessary to follow the same

magnetic journey in the testing which is to be taken in the damping experiments.

If the distance of the centre of the auxiliary magnet from the centre of the needle is d_0 centimeters, and if l_0 is the half magnetic length of this magnet, the moment of the couple which it exerted upon the needle before the latter was deflected, and which just balanced the moment due to the magnet to be tested, was $\frac{M_0 d_0}{(d^2_0 + l^2_0)^{\frac{3}{2}}}$.

When the magnet under the test is removed, the needle deflection (δ_0) caused by the auxiliary magnet alone is usually too large to be easily measured by aid of the telescope and scale; but if this magnet be removed on its track to such a distance that the deflection δ' can be determined, and if the distance between the centres of the magnet and needle is then d',

$$\frac{\tan \delta_0}{\tan \delta'} = \frac{d_0}{(d^2_0 + l^2_0)^{\frac{3}{2}}} \cdot \frac{(d'^2 + l^2_0)^{\frac{3}{2}}}{d'}, \tag{30}$$

and M' can be determined in terms of M by means of the equation

$$\frac{M' - M}{M} = \frac{\tan \delta}{\tan \delta_0}. \tag{31}$$

VII. The first magnet (Q′) used with this apparatus was about 4.0 centimeters long and weighed about 7 grams. The whole suspended system had a moment of inertia in vacuo almost exactly equal to 43.0, and the magnetic moment of the seasoned magnet (Q′) when placed with its axis perpendicular to the meridian was about 29.8 units. Its induction coefficient under these circumstances was about 0.0242; its moment in a field of 10.37 gausses was 38.7. Most of the records were made with the drum revolving very slowly at the rate of a turn in 348 seconds: the normal length of a record was 479 millimeters. The periodic time of the swinging system varied from 50.8 seconds to 1.20 seconds in the fields actually used. The torsion coefficient of the fibre was under all circumstances here considered much too small to be appreciable.

Figures 5 (Plate 2) and 6 (Plate 3) represent oscillations of the suspended system of which the magnet Q' was a part under fields of about 2 and 12 gausses respectively. In the case shown in Figure 7

(Plate 3) the magnet was deflected through an angle of perhaps 10° and then suddenly released. The record begins at the point O, where a nearly straight line indicates that the magnet was on its way through the position of equilibrium and out on the other side to a point corresponding to a deflection of about 2.5°, after which the amplitude decreased gradually and regularly. The field here was about 19.3 units. Figures 8, 9 (Plate 4) show the effect of suddenly applying a comparatively strong field (14.3) gausses when the system is already swinging in a field of about 2 units.

The curious irregularity in the spacing of the record in the last diagram after the strong field was applied came from the fact that the

TABLE VIII

PERIODIC TIME, 13.2 SECONDS

Successive Amplitudes

Measured	Computed	Measured	Computed
857	857	302	283
763	760	269	249
680	673	239	222
605	600	213	195
539	533	189	172
480	480	169	160
427	423	150	148
380	374	134	136
339	325		

magnet was making oscillations in a vertical plane with an amplitude of about 2′. When the system was at rest, the axis of the magnet and the axis of the solenoid were in the same vertical plane but differed from each other in direction by a small fraction of one degree.

To illustrate the fact that in a weak field where the period of the oscillation is long the amplitude of the motion decreases regularly with a practically constant decrement, and that in somewhat stronger fields the departure from this law is nearly inappreciable, except perhaps at the very beginning of the motion, two or three sets of typical measurements will serve. In very strong fields, when the initial deflections are fairly large, the motion cannot be explained with any good approximation to accuracy on the assumption that the air resistance furnishes a couple proportional to the angular velocity.

When the periodic time was as short as 1.2 seconds, a curve of the

family $A.e^{-at}$ which passed through the crests of the figure at the middle of the diagram fell distinctly below the crests at the beginning.

From measurements of photographic records taken with Q' for eight different values of the current in the solenoid, the period (T), the damping coefficient ($2a$), the logarithmic decrement (λ) were determined for every case; the intensity of the magnetic field (H) about the magnet was then found by adding the original strength of the field to that caused by the measured steady current in the solenoid,

TABLE IX

PERIODIC TIME, 5.20 SECONDS

Successive Amplitudes

Measured	Computed	Measured	Computed
795	789	390	388
744	740	365	364
696	693	343	341
655	650	320	320
611	610	302	300
574	571	285	281
536	536	266	264
504	502	250	248
474	471	235	232
445	441	220	217
418	417	205	197

and a fairly approximate value of the moment of the magnet was computed from the H thus found and the results of measurements made with the magnet in the induction coefficient apparatus described above. When these quantities were known, it was comparatively easy to determine β from the equation $(\pi^2 + \lambda^2)/T^2 = \beta^2$, and then to get an approximate value of the apparent moment of inertia of the swinging system from the formula $I = MHT^2/(\pi^2 + \lambda^2)$. Some of the results obtained by studying many records of the motion of this suspended system are given in the next table.

TABLE X

Period	M H	Damping Coefficient	Logarithmic Decrement
15.90	6.38	0.00914	0.0726
13.8	9.86	0.00927	0.0640
9.9	16.64	0.00985	0.0487
8.05	24.2	0.01029	0.0414
7.63	29.9	0.01067	0.0407
2.85	213	0.01467	0.0209
2.18	359	0.01651	0.0180
1.12	1418	0.01907	0.0115

As has been said above, it is possible to obtain from these data values for the apparent moment of inertia of the oscillating system, but since a slight change in any one of several of the quantities measured might introduce a great change in the quantity computed, the results must be considered rough. Such a change in the intensity of the earth's field as might come from a passing train of electric cars at two hundred yards distance would appreciably affect the first value given.

The results of this computation are respectively 161, 188, 163, 157, 174, 173, 171, 178. So far as one may judge from these and from similar sets obtained with other systems there is no very strong evidence that I changes materially with T, unless it be for extreme values. The damping coefficient is by no means constant, for its value increases rapidly with the restoring force but not according to any easily recognizable law.

VIII. In the next series of experiments with the apparatus represented in Figure 6 Q' was displaced by another small bar magnet 6.0 centimeters long which, when placed perpendicular to the earth's field at room temperature, had a magnetic moment of 101.2 units. This new magnet (Q'') had a moment 129.5 in a field of 9.07 gausses, and a moment 140.2 in a field of 19.93 gausses, when the field was slowly increased. The same mica vane (x) was used as in the work with Q'.

The results of measurements made upon photographic records made with fields of seven different strengths appear in the next table.

TABLE XI

Period	M H	Damping Coefficient	Logarithmic Decrement
14.53	12.5	0.0094	0.0683
6.31	68.0	0.0120	0.0379
4.47	123	0.0136	0.0304
2.97	270	0.0158	0.0235
1.81	669	0.0196	0.0177
1.23	1808	0.0222	0.0137
0.81	4396	0.0283	0.0114

At another time a long series of observations were made with the same system, under somewhat different initial circumstances of field and perhaps of moisture in the atmosphere, with the results given below.

TABLE XII

Period	I	Period	I
12.30	287	1.60	285
10.44	295	1.27	295
7.90	285	1.13	297
3.29	280	0.98	293
2.38	280	0.81	292
1.92	294		

Here again the apparent moment of inertia is nearly constant but the damping coefficient increases rapidly as the field about the magnet becomes more intense.

Many kinds of physical measurements concern themselves with the behavior of oscillating systems, and it is often necessary to determine what the apparent moment of inertia of a system is if the motion is in air, and what the exact value of the damping coefficient is at any time. If this is not constant throughout the whole motion, — as it should be if it follows the Gaussian law, which assumes the existence of a fixed logarithmic decrement, — it is necessary to find out how it varies with period and amplitude. If one uses a d'Arsonval galvanometer to measure changes of magnetic flux in a large mass of iron, and for reasons of sensitiveness at some point of a hysteresis diagram needs to introduce extra resistance into the circuit or to remove some which is there already, one cannot compute the effect of the change unless one knows, not the real, but the apparent, resistance of the galvanometer coil, and this depends upon the "constants" of the motion which must be determined with some care; it would not be difficult to show that such deviations from the Gaussian law as one frequently encounters in practice need to be carefully taken into account in accurate work. The fact that the swinging system comes to rest in a comparatively short time suggests that the law may not be exactly followed at any part of the motion.

If, then, a swinging magnet or galvanometer coil is exposed to a relatively strong air damping, we must expect that unless the amplitude is very small there will be an appreciable departure from the Gaussian law. If the system be turned out of the position of equilibrium through a considerable angle and then released, it moves rapidly through this position and out on the other side to a new elongation

corresponding to a displacement much smaller than the one from which it started; and this modifies profoundly the theories of some ballistic instruments, but after this the subsequent decrease of the amplitude takes place slowly and regularly, accompanied usually by a slowly decreasing logarithmic decrement. For any small number of swings after the first few, however, the constancy of the logarithmic decrement can often be assumed with sufficient accuracy for ordinary purposes.

The moment of inertia of the swinging system cannot as a rule be computed with any fair approximation from a knowledge of the masses and the geometrical dimensions of the bodies of which the system seems to be made up, for a comparatively large mass of air accompanies the visible system and materially increases the inertia. The apparent moment of inertia of the system seems usually to remain practically unaltered when the moment of the restoring couple which dominates the swings is changed within wide limits, but under these circumstances the coefficient of damping generally increases rapidly as the restoring moment is increased, and the period decreases. If the restoring moment is due to an external field the periodic time remains fairly constant as the amplitude decreases; but if the moment comes from the torsional rigidity of a stiff wire, the period frequently lengthens somewhat as the amplitude grows small.

In case of a d'Arsonval galvanometer coil hung by different pieces of gimp or wire successively, the damping coefficient is practically the same for large differences of period if the resistance of the coil circuit is unchanged; but if this resistance is changed, the damping coefficient changes in a manner to be quantitatively explained by assuming that the coil has an apparent resistance larger than its real resistance. This apparent resistance may be considered as a constant of the coil as long as the level of the instrument is unchanged. If the righting moment of a swinging coil or magnet exposed to air damping is weak and comes from the torsional rigidity of a piece of fine gimp or fibre, the motion often seems to be anomalous because it depends upon obscure elastic changes.

X

ON THE MAGNETIC BEHAVIOR OF HARDENED CAST IRON AND OF CERTAIN TOOL STEELS AT HIGH EXCITATIONS[1]

During the last few years the use of hardened cast iron for permanent magnets has increased very much, and this material has proved especially useful for such shapes as could not be easily forged from steel without heating the metal red hot a number of times and thus making it magnetically unsatisfactory. Cast-iron magnets are very cheap, and they may be made quite as strong and as permanent as magnets made of the best tool steel, even if in strength, though not in permanence, they fall a little behind magnets made of special "magnet steels." Moreover, and this is sometimes of very great importance, the temperature coefficient[2] of a seasoned cast-iron magnet is usually much smaller than that of a magnet of the same strength made of forged or formed steel. This paper discusses briefly a number of determinations of the permeability of specimens of fairly soft and of glass-hard cast iron of the same kind, under excitations up to about 15,000 gausses,[3] and, for purposes of comparison, considers also some measurements made upon hard and soft Stubs "Polished Drill Rod" and upon hard and soft "Crescent Polished Drill Rod."

The principal apparatus used consisted of a yoke (Figure 1) which weighed about 300 kilograms and was excited by a current (from a storage battery) running through a set of amperemeters in series with the coil of 2956 turns wound on the spools shown in the diagram. The yoke was furnished with a number of pairs of pole pieces or jaws, to receive specimens of different lengths and shapes. To measure the amount of the induced current in a test coil wound closely upon the

[1] *Proceedings of the American Academy of Arts and Sciences*, March, 1909, vol. xliv, no. 13.

[2] Peirce, *ibid.*, vol. xxxviii, 1903; vol. xl, 1905.

[3] Rowland, *Phil. Mag.*, vol. xlvi, 1873; Fromme, *Ann. d. Phys.*, vol. xiii, 1881; vol. xxxiii, 1888; Stefan, *Ann. d. Phys.*, vol. xxxviii, 1889; H. E. J. G. DuBois, *Ann. d. Phys.*, vol. xxxi, 1888; vol. li, 1894; B. Walter, *Ann. d. Phys.*, vol. xiv, 1904; Ewing, *Magnetic Induction in Iron and other Metals;* DuBois, *The Magnetic Circuit in Theory and Practice.*

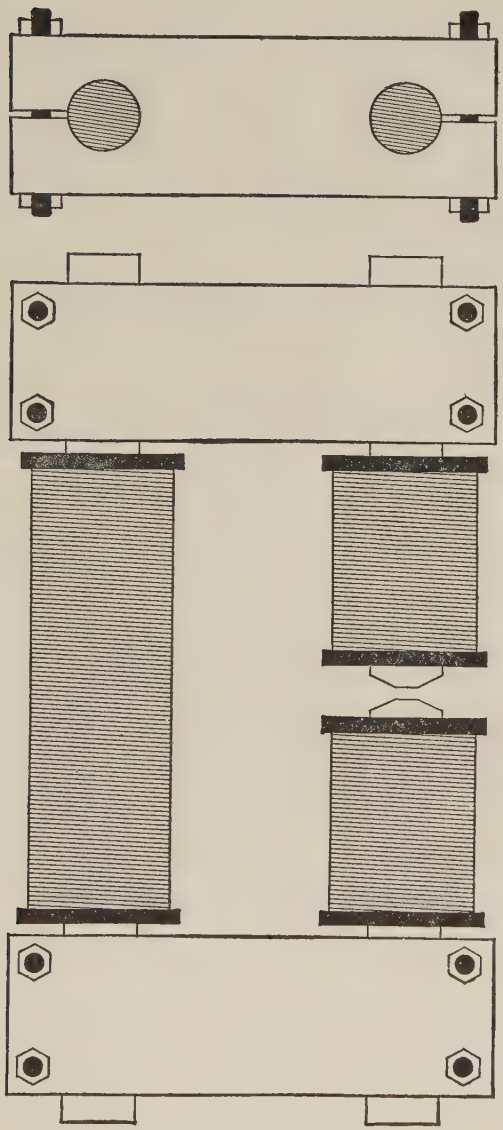


FIGURE 1

piece to be examined, a ballistic galvanometer (V), described in a former paper,[1] was used. The period of this instrument was so long that the throw due to a reversal of the exciting current of the yoke did not appreciably differ from the throw which the same quantity of electricity would have caused if it had been sent instantaneously through the circuit.

The specimens used in the work here described were of two forms. The first form (C, Figure 2) was a cylinder about 1.27 centimeters in diameter and about 15 centimeters long over all, with tapered ends to fit tightly in sockets in the ends of the conical pole pieces of the

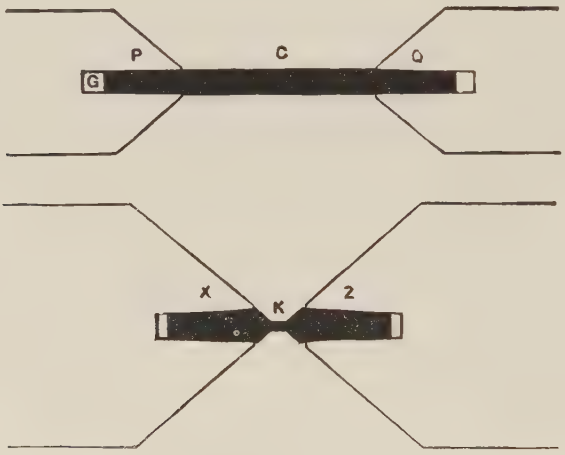


FIGURE 2

yoke. The sockets (G) were first turned out in the lathe and then finished by a reamer made and ground by the machinery afterwards used to cut the tapers on the ends of the test pieces. Each test piece of the hard cast iron had first to be ground to the form of a true cylinder in a universal grinding machine and then to be tapered off at the ends with the help of a centre grinder, mounted motor and all, in the tool post of an engine lathe. All the work was done by Mr. G. W. Thompson, the mechanician of the Jefferson Physical Laboratory, in the most skilful manner, and the reluctance of the joints must have been relatively very small. When a specimen of this shape was in posi-

[1] *Proceedings of the American Academy*, vol. xliv, 1909.

tion between the pole pieces of the yoke, and a steady current of at least two or three amperes was passing through the exciting coil, it was assumed that the value of H within the small cylinder (C) near the middle of its length was the same as the value of H in the air just outside the metal. The ground of this assumption was a series of experiments carried out some months ago. A piece of homogeneous steel rod about half an inch in diameter and about three hundred and fifty diameters long was placed within a solenoid consisting of 20,904 turns of thoroughly insulated wire wound on a straight piece of stout brass tube about an inch in inside diameter and rather more than sixteen feet long. Near the middle of the steel rod a test coil of fine insulated wire was wound closely on it, and then, with its leads, made thoroughly waterproof, so that a current of tap water could be kept running around the rod in the brass tube to hold the temperature of the steel nearly constant when strong currents should be sent through the solenoid. The steel was first demagnetized by means of a long series of currents in the solenoid, alternating in direction and steadily decreasing in intensity, and then a series of steady direct currents of carefully measured intensities, each a little stronger than the last, were sent through the solenoid and reversed many times at each stage to determine the corresponding value of B in the steel. In this manner it was possible to get a satisfactory curve of ascending reversals for the steel up to H = 400 and B = 20,500, nearly. The length of the rod was, relative to its diameter, so great that the demagnetizing factor was very small and the correction for the ends very easily made. The rod was then demagnetized again, and the process described was repeated two or three times until the resulting table of B versus H values seemed to be well determined. After this, short pieces of various lengths, cut from the rod which had been tested, were used in the yoke and were mounted in different ways in the hope of discovering some satisfactory method of studying the permeability of the steel by experiments upon these pieces, which should give the same results up to an induction of about 20,000 as those already obtained by the work with the long solenoid. After long trial, a length of cylinder was found which seemed, in this particular yoke, to make the values of H at the centre of the length of the specimen practically the same as the value in the air just outside the metal. Two different

materials were used in stout rod form in the long solenoid, Bessemer Steel and "Compressed Steel," an extremely homogeneous kind of steel prepared for us by the Boston Compressed Shafting Company.

In all the cases tried specimens of the size and shape described above seemed to give the same permeability up to values of the induction as great as 20,000 as the long solenoid did, and, for somewhat

TABLE I

Cylinder of Soft Cast Iron

H	B	I	μ
114	9950	782	87.3
172	10800	846	62.8
433	13900	1070	32.1
744	15750	1200	21.2
1234	17300	1280	14.0
1820	18170	1300	10.0

higher values of B, to yield results which agreed with those obtained, where it first becomes trustworthy, by the "Isthmus Method."

After the central portion of each of these specimens had been covered with an extremely thin coat of varnish, the diameter was determined under the microscopes of a Zeiss Comparator, reading to the nearest thousandth of a millimeter directly. Then two test coils, each of twenty turns of very fine, well-insulated wire, were wound side by

TABLE II

Isthmus of Soft Cast Iron

H	B	I	μ
12700	31100	1465	2.5
13550	32100	1475	2.4
13800	32500	1488	2.4
15100	33650	1472	2.2

side in a single layer over the varnished metal and extended over perhaps a centimeter at the middle of the rod. These coils were tested against each other when the specimen was in the yoke, to see if they were alike, and if they were, both, in series, formed the inner test coil (L) to be used in the measurements. The second testing coil (M) was wound on a very carefully made spool of boxwood which had been seasoning for many years. This spool kept its diameter practically unchanged during the measurements here recorded, though it

shrank very slightly soon after it was first made. The diameter of the wood was about 1.9135 centimeters, and that of the outside of the wire of the coil about 1.9591 centimeters, the last figure in each case being, of course, doubtful. Hard rubber is so susceptible in a magnetic field as to make it impossible to use a spool of this material to sup-

TABLE III

Cylinder of Hard Cast Iron

H	B	I	μ
142	7860	614	55.4
254	9700	752	38.2
339	10850	836	30.6
684	13050	983	19.1
915	14050	1044	15.4
1570	15900	1138	10.1
2020	16800	1176	8.3

port a testing coil. When the specimen was in place between the jaws of the yoke, it was covered by the shorter spools of the yoke.

The value of H in the air just outside the metal was obtained by reversing the exciting current of the yoke when L and M were opposed to each other in the circuit of the ballistic galvanometer (V) described above. When L alone was used in the galvanometer circuit, and proper corrections for the air lines through L had been made by the

TABLE IV

Isthmus of Hard Cast Iron

H	B	I	μ
10900	26540	1245	2.4
13200	28600	1226	2.2
14800	30200	1226	2.0

use of the H just determined, it was possible to measure the induction flux in the metal.

The second kind of specimen shown approximately by K, Figure 2, was of the shape usually employed in isthmus measurements. Cast iron differs from steel in that it can be heated so hot before it is chilled that it becomes eventually hard throughout its mass, while steel can be hardened only for a little distance from the surface. On the other hand, it is not easy to harden a long slender rod of cast iron without its becoming slightly crooked in the process. An isthmus piece of

cast iron has, therefore, to be ground into shape at much labor, from a glass hard cylinder. The hardened steel isthmus pieces, on the contrary, were shaped while soft, and were then chilled inside a supporting tube after they had been heated in a gas furnace.

TABLE V

Cylinder of Soft Crescent Drill Rod

H	B	I	μ
122	13060	1030	107.0
209	16730	1315	80.1
272	17190	1351	63.2
486	18400	1425	37.9
783	19150	1462	24.5
1535	20600	1516	13.4
1798	20900	1527	11.6

TABLE VI

Isthmus of Soft Crescent Drill Rod

H	B	I	μ
4860	24600	1570	5.1
7190	27100	1584	3.8
10000	29700	1569	3.0
12020	32500	1629	2.7
13150	33800	1642	2.6

TABLE VII

Cylinder of Soft Stubs Polished Drill Rod

H	B	I	μ
132	14600	1154	110.6
299	16700	1307	55.8
540	18100	1395	33.5
830	19000	1445	22.9
1380	20200	1495	14.6
1780	20800	1514	11.7

The "Isthmus Method" for determining the permeability of a small piece of magnetic metal at a high excitation rests, of course, upon the assumption that the value of H just without the test piece is equal to the average value of H over the cross section of the metal at the neck. At the surface of a magnet the tangential components of the magnetic force are continuous, while the normal component is discontinuous: it seems desirable, therefore, before one applies the method in any particular case, that one make sure that the lines of the field in the air space to be used are practically straight and parallel

to the axis of the specimen. Any person who has had experience in using large yokes at high excitations, where because of the low permeability of the metal the leakage is very great, knows how slight a change in the shape of a specimen may alter the field in the neighborhood of the test piece very sensibly. An isthmus piece of steel which

TABLE VIII

Isthmus of Soft Stubs Drill Rod

H	B	I	μ
7900	26800	1500	3.4
13850	33200	1545	2.4
14900	34400	1554	2.3
15800	36200	1570	2.3
17100	37000	1587	2.2

TABLE IX

Cylinder of Hard Crescent Drill Rod

H	B	I	H	B	I
114	8600	677	850	14650	1097
175	10050	786	1041	15200	1127
254	11300	879	1337	15950	1162
503	13000	993	1894	17000	1200

had been hardened unequally might warp the field sufficiently to make the observations of the permeability wholly erroneous.

After much consideration I have decided not to print the results of my measurements upon isthmus pieces of glass-hard Stubs and Crescent Drill Rod for the reason that the maximum values of I seem to be rather too high. In one case, indeed, the effect of hardening an

TABLE X

Cylinder of Hard Stubs Drill Rod

H	B	I	H	B	I
123	8600	675	564	13750	1049
180	10020	783	982	15350	1143
256	11300	878	1416	16250	1216

isthmus piece of steel was to make the ultimate value of I rather greater than before, though for moderate excitations the permeability was less. I hope to try soon the effect upon the uniformity of the field about the isthmus of harder jaws. The results obtained with the hard cast iron seem to be good.

The cast iron used for the observations recorded below, which was extremely soft and easy to work, came from the Broadway Iron Foundry of Cambridgeport, Mass., where we have obtained during the last few years a large number of castings of different forms for permanent magnets which proved when made and seasoned to be very strong and to have remarkably small temperature coefficients.

It will be noticed that this iron while soft is rather more permeable than that which was the foundation for the formula for reluctivity in "Ordinary Dynamo Cast Iron" given by Messrs. Houston and Kennelly in their Electro-Dynamic Machinery, but is very similar so far

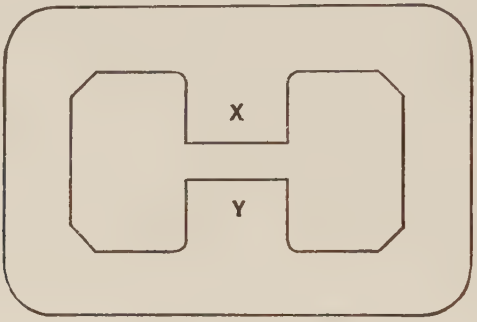


FIGURE 3

as results are available with the standard "Gray Cast Iron" used for the table given in the pamphlet on the "Magnetic Circuit" of the International Textbook Company. Although I had at command a much larger yoke than the one used, no attempt was made to carry the excitation beyond 15,000 Gausses. The ultimate value of I in my hardened cast iron was about the same as that which Ewing gives for "Cast Iron" in "Magnetic Induction in Iron and Other Metals," § 93.

The magnetic effects of hardening upon a mass of cast iron are often very noticeable at comparatively low excitations. The two halves of each of two thick castings, one soft, the other very hard, of the form shown in Figure 3, were wound with 156 turns each of insulated wire, and the two coils on each casting were so connected in series that when a current was sent through the circuit both conspired to make one of the projections (say X) a north pole and the other (Y) a south

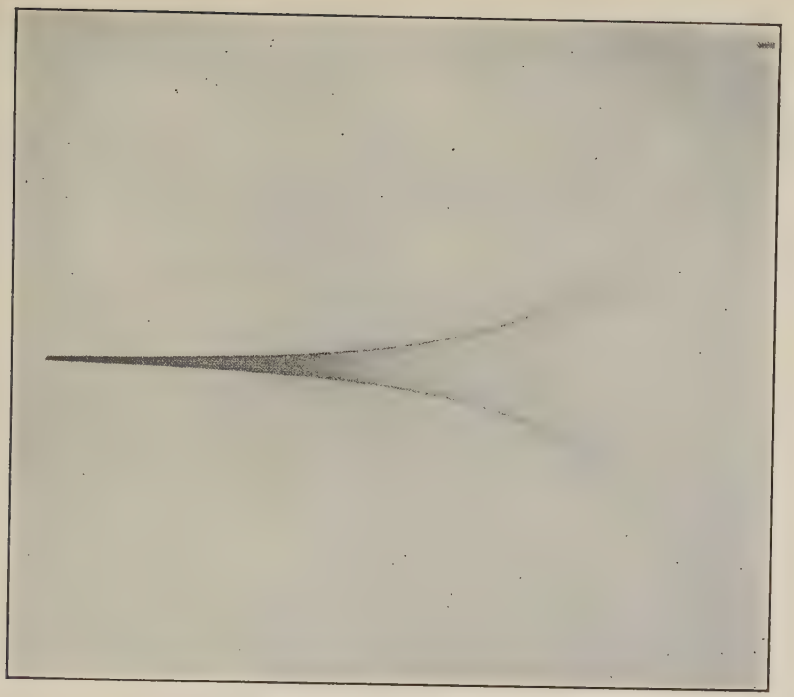


FIGURE 1

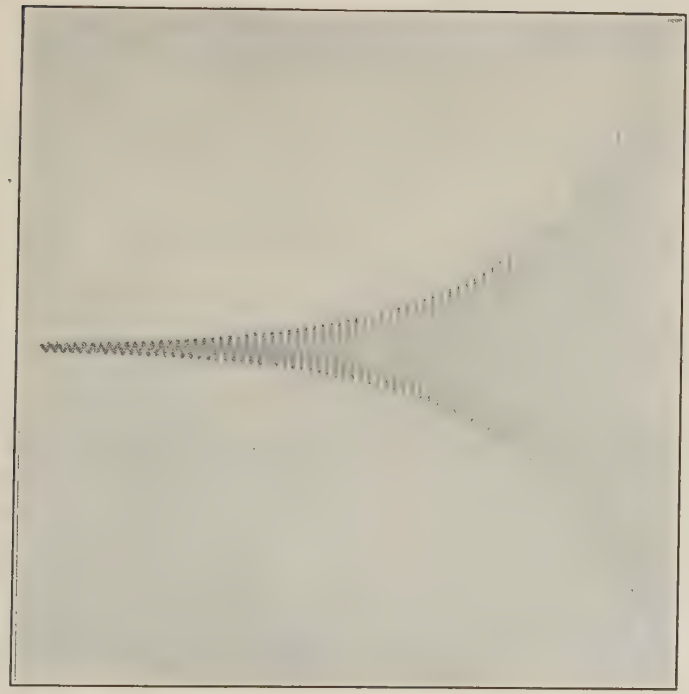
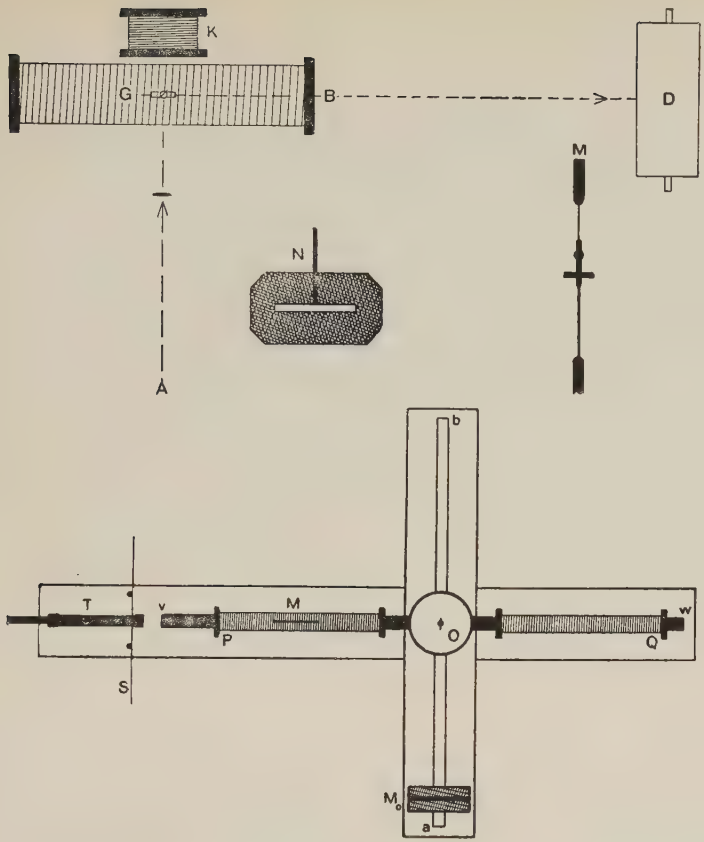


FIGURE 2



FIGURES 3 AND 4

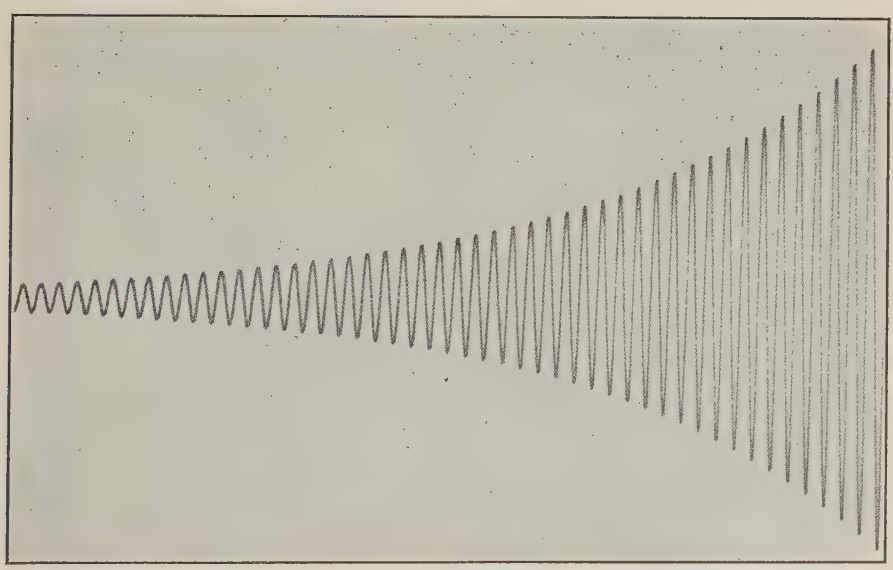


FIGURE 5

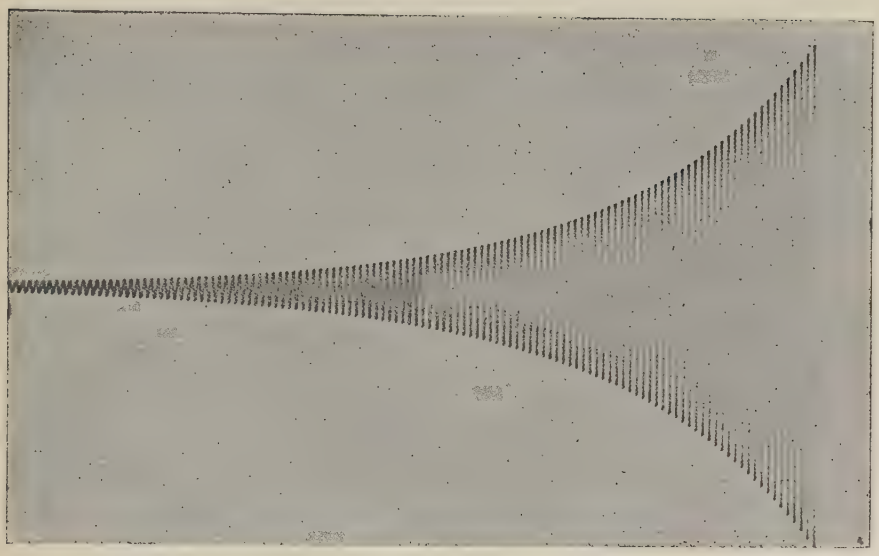


FIGURE 6

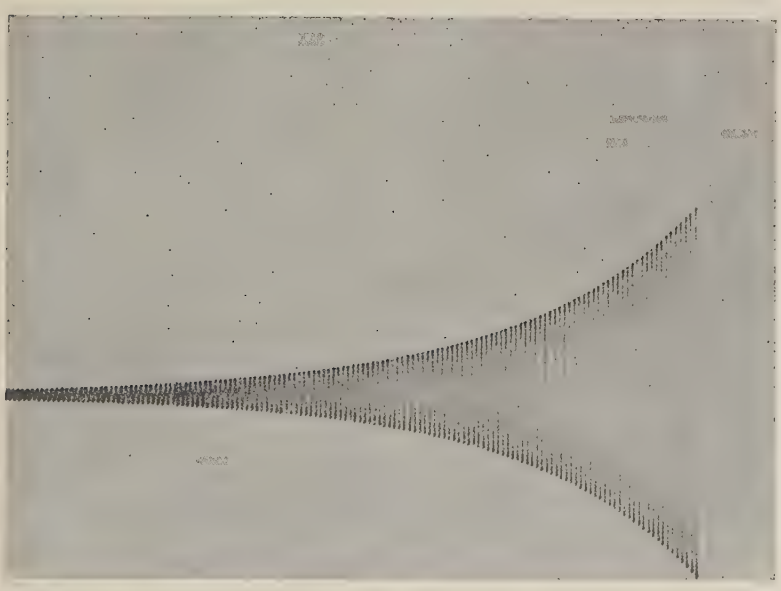


FIGURE 7

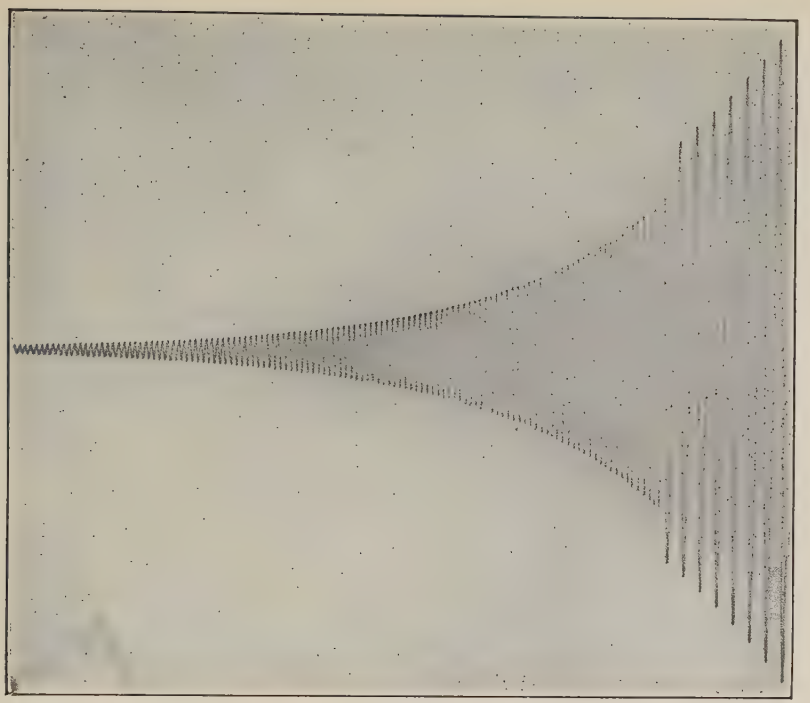


FIGURE 8

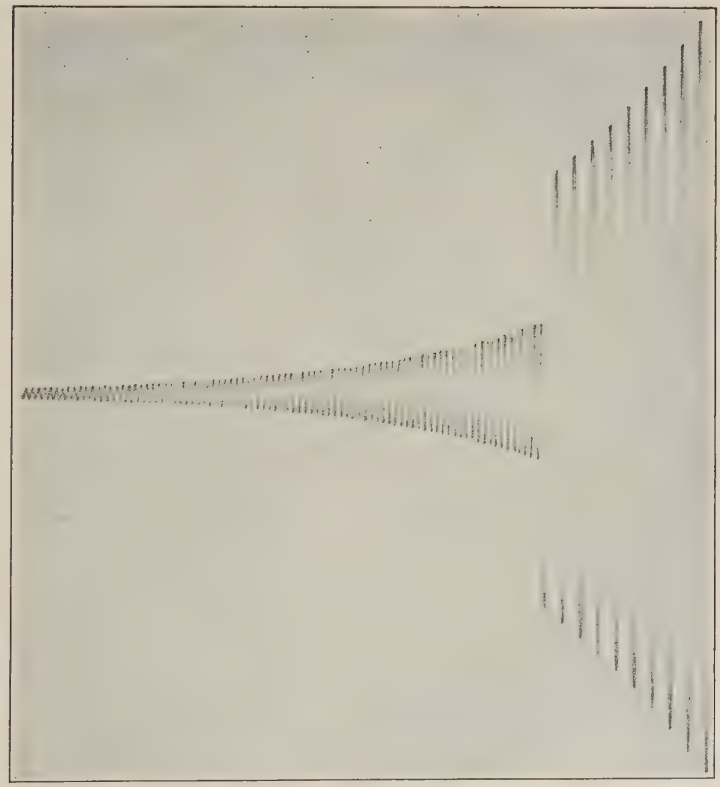


FIGURE 9

pole. With each of the castings a rude kind of hysteresis diagram was obtained by measuring for different current strengths the values of the induction flux across a definite area in the air gap between the poles. These fluxes plotted against the corresponding currents gave the diagrams shown in Figure 4.[1] The A curve belongs to the soft casting, the B curve to the hard one. While it would be difficult to explain the exact meaning of these curves in terms of the permeabilities of the iron, the differences are striking.

It appears from the observations of Ewing upon Vicker's Tool Steel that in the case of the specimen he used the value of I was still rising, and at a fairly rapid rate, when H grew to be as great as 14,000. The same tendency, it will be noticed, is shown very clearly in the two kinds of steel which I have studied. These were chosen as being perhaps the best annealed brands of fine tool steel to be had in the market.

The very interesting results given in Table XI were obtained by Mr. John Coulson, who has helped me in all this work, with a standard

TABLE XI

Cylinder of Jessops Round Tool Steel

H	B	I	H	B	I
110	15250	1205	960	19950	1510
158	16200	1280	1030	20100	1520
255	17450	1370	1200	20450	1530
500	18850	1460	1680	21100	1545
645	19100	1470	1980	21600	1560
810	19700	1505			

cylinder, 1.283 centimeters in diameter, made of Jessops Tool Steel. This celebrated brand of steel seems harder under the file than the Stubs or the Crescent Drill Rod, but is remarkably permeable, and has been much used for permanent magnets.

My thanks are due to the Trustees of the Bache Fund of the National Academy of Sciences who have kindly lent me some of the apparatus used in making the observations described in this paper.

[1] Identical with Fig. 8 on p. 48.

XI

THE THEORY OF BALLISTIC GALVANOMETERS OF LONG PERIOD [1]

If a ballistic galvanometer is to be used to measure the whole quantity of electricity which flows impulsively in a circuit when a condenser is discharged through it, or when the flux of magnetic induction through the circuit is suddenly changed, it can generally be assumed that the time during which the current lasts is so short that the flow practically ceases before the suspended system of the instrument has moved sensibly from its position of rest. That is, that the whole time of flow is not greater than, say, one fiftieth part of the time required for the needle or suspended coil to reach the end of its throw.

It is often desirable, however, to determine with accuracy the change of magnetic flux in a massive closed iron frame caused by a given change of excitation, and in such a case it usually happens that eddy currents in the metal or the inductance of the exciting coil so retard the change that the process lasts for a number of seconds at least. Under these circumstances a ballistic galvanometer of any ordinary form is practically useless. Indeed, according to the experiences [2] of DuBois with such galvanometers as are to be found in most laboratories, the ballistic method fails when the time required for the change exceeds about one second.

Slow flux changes can be measured, nevertheless, with the aid of photographic records from a suitable oscillograph [3] either in the main circuit of the magnet or in the circuit of a testing coil wound about the iron. My experience with hundreds of such records seems to show, however, that the thickness of the photographed line obscures some-

[1] *Proceedings of the American Academy of Arts and Sciences*, March, 1909, vol. xliv, no. 11.

[2] DuBois, *The Magnetic Circuit in Theory and Practice*, Atkinson's translation, § 216, London, 1896.

[3] T. Gray, *Phil. Trans.*, vol. clxxxiv, 1893; Thornton, *Electrical Engineer*, xxix, 1902; *Phil. Mag.*, vol. viii, 1904; *Electrician*, 1903; Peirce, *Proceedings of the American Academy*, vol. xli, 1906; vol. xliii, 1907.

what the slow changes when the exciting current has nearly reached its new value, and in the very sensitive instruments sometimes required for use in a secondary circuit there is a small but occasionally troublesome lag just at the beginning of the motion. For all ordinary purposes this method is wholly satisfactory if not always easy or convenient to carry out.

Such fluxmeters as I have been able to procure, though admirable in many ways, have not been so free from crawling, due apparently to the paramagnetic properties of their copper coils, that their indications can be trusted for very slow magnetic changes. If the fluxmeter coil [1] is not wound on a metal frame, the mutual damping caused by the action of currents in the coil, and the core which it surrounds, are not always effective unless the resistance of the exterior circuit is small, and this frequently makes an instrument which works very well for one piece of work, nearly useless for another.

When the excitation of the core of a large electromagnet initially in a given magnetic condition and under a given excitation is changed by a predetermined amount, it sometimes happens — as is well known — that the resulting change in the magnetic flux through the iron depends somewhat upon the manner in which the exciting current is changed; that is, the flux change is different when the current in the magnet coil is changed very gradually or in short steps from what it is when the change is made very suddenly. This difference is generally small, and seems to depend upon a variety of circumstances [2] in a way not yet very well understood, but it must be determined for every large magnet if the behavior of the core under given conditions is to be predicted with any great accuracy.

I have recently had occasion to inquire how the changes of magnetic flux in each of a number of large cores, of which two are represented by Figures A and B,[3] corresponding to given changes in the current in the exciting coil, depend upon the manner of growth of that current, and since such oscillograph records as I was able to make were not

[1] Beattie, *Electrician*, December, 1902; Peirce, *Proceedings of the American Academy*, vol. xliii, 1907.

[2] G. Wiedemann, Galvanismus, vol. iii, 738; Gumlich and Schmidt, *Electrotechnische Zeitschrift*, vol. xxi, 1900; Ruecker, *Inaugural Dissertation*, Halle, 1905; A. Hoyt Taylor, *Phys. Rev.*, vol. xxiii, 1906.

[3] Figs. A and B identical with Figs. 15 and 1 on pp. 122 and 51 respectively.

wholly satisfactory for the purpose, I found it desirable to attempt to procure a ballistic galvanometer (preferably of the d'Arsonval type, to avoid disturbances due to changes in outside magnetic fields) of period so long that the throw of the coil due to a change of flux of the usual sort, lasting for say thirty seconds, should not be sensibly different from the throw due to the same amount of electricity sent impulsively through the coil when at rest in its position of equilibrium.

The galvanometer I sought did not need to be very sensitive, but it must have one property which, according to my experience, is rare in suspended coil instruments; that is, there must not be the slightest sensible shift of the zero point due to thermal currents or to chemical action at the junctions when the galvanometer circuit should be closed on itself. This condition forbade the leading of the current into the galvanometer coil through the phosphor bronze or steel gimp by which the coil was suspended, and required that the whole galvanometer circuit, even to the binding posts and connectors, should be of one metal, copper.

It is of course not desirable to make the period of a ballistic galvanometer long by making the righting moment due to the suspending fibre small, for a weak fibre makes the zero point uncertain, and a large throw on one side usually shifts the zero point slightly in that direction unless the gimp is even stouter than that commonly used in sensitive instruments. It seemed necessary, therefore, to increase the moment of the suspended system so much that in spite of a stiff suspending gimp the period should be long.

In the case of a galvanometer coil with a period several minutes long, it is difficult to tell by mere inspection for a few seconds whether the coil is really at rest at its zero or whether it has a very slight velocity which in the course of its slow swing will lead to an addition of two or three millimeters to the amplitude of the throw. For this reason it was desirable that the coil should be subjected to some slight electromagnetic damping, though, as will appear later on, it was not possible to damp the coil critically.

The requirements enumerated above are so simple that it seemed at first that there would be no difficulty in meeting them all, and this would have been the case if it were not for the fact that the best copper and silver wire, and the best copper, silver, and aluminium sheet

to be had in the market are usually so highly paramagnetic that in an intense magnetic field the galvanometer coil and the metal frame on which it is wound, if a frame be used, often acquire a large magnetic moment, and this increases in an irregular way the righting moment of the suspended system — perhaps to many times the value due to the gimp alone. The difficulty is an old one; many persons have struggled with it, and some have succeeded in overcoming it more or less completely, by great care in the preparation of special wire for the purpose. The difficulties are, however, very much increased when it is necessary to provide a sufficient electromagnetic damping (air damping is sometimes objectionable) for a suspended system which in order to have the requisite moment of inertia must weigh perhaps 300 grams. Silk insulating material is generally magnetic, and so is most paraffine wax. A certain closed frame made by Mr. G. W. Thompson, the mechanician of the Jefferson Physical Laboratory, of the best obtainable sheet copper, to hold the coil of a d'Arsonval galvanometer of the common cored type, had a period of oscillation of about 2 minutes when suspended by a certain piece of gimp in free space, but a period of only 9 seconds when put in place in the instrument. In this case the righting moment due to the fibre was clearly wholly overshadowed by that due to the magnetism of the copper. When copper was wound on this frame, the magnetic moment of the whole, if placed between the poles of the permanent magnet, became so large that the whole suspended system could be deflected at will, when the circuit was open, by a bar magnet held in the hand outside the frame of the instrument.

It is easy to make the period of an ordinary d'Arsonval galvanometer of the Ayrton and Mather form as long as, say, 120 seconds, by attaching two small brass masses symmetrically to the ends of a horizontal aluminium wire centered on the axis of suspension of the coil (Figure C),[1] though it is not always easy to balance the coils and its weights so exactly that the throws shall be symmetrical on both sides of the zero point and that the instrument shall not be unpleasantly affected by changes of level. Galvanometers of this kind are often useful: several (one with a period of 158 seconds) have been used for years in the Jefferson Laboratory, and Professor A. Zeleny has lately

[1] Identical with Fig. 35 on p. 141.

employed a loaded coil galvanometer in his investigations of the properties of condensers. When the case of a d'Arsonval galvanometer is large enough, it is obviously better to load the coil with a disk centred on the axis of suspension than by several small masses, and in the instruments to be described in this paper thin disks with strongly reinforced rims were employed.

Two loaded coil d'Arsonval galvanometers have been constructed for me by Mr. Thompson. The first (V), shown in Figure 1, Plate 5, is about 76 centimeters high over all, and the gimp by which the coil is hung is 32 centimeters long. The brass disk, which is 11.4 centimeters in diameter, is rigidly attached to the rectangular frame (3 centimeters $\times$ 7 centimeters) upon which the copper wire coil is wound, and is accurately perpendicular to the axis of suspension.

After the copper frame constructed for this instrument had proved unsatisfactory, a cast type-metal frame was made to take its place. Of course this frame is not nearly so effective in damping the swings of the coil as a copper frame would be, but, on the other hand, its magnetic moment when it lies between the poles of the magnet of the galvanometer is not large. The insulated copper wire on the frame, however, gives a comparatively high moment to the whole suspended system, and the period of the galvanometer is much shorter — only about 140 seconds — than we supposed it would be with so large and heavy a disk. The binding posts and all the other connections are of copper, and the current is led into and out of the coil by two copper spirals under the disk, so fine that they do not exert any appreciable righting moment when the coil is deflected. The gimp is of steel, just stout enough to hold up securely the loaded coil.

The second galvanometer (W), represented in Figure 2, Plate 5, is about 111 centimeters high over all and 30 centimeters in diameter; the suspension gimp is about 80 centimeters long. It seemed nearly hopeless to attempt to get a sufficiently small righting moment with a hollow coil made of such wire as was to be obtained in the open market, so a coil of the Ayrton and Mather form was made for this instrument. The disk is accurately mounted on a metallic rod upon which the coil is fastened. The disk is built up of a thin sheet of flat aluminium with a brass rim about 24 centimeters in outside diameter and 15 millimeters in width. The current enters and leaves the coil

through very fine copper spirals, one above and one below. If No. 44 or No. 46 copper wire be rolled out flat between jewellers' rolls or other similar device the resulting gimp serves to make a spiral which has extremely little torsional rigidity. It is possible to increase the number of field magnets in this instrument at pleasure. The logarithmic decrement of the galvanometer is small, but it has proved to be possible to bring the coil to rest at its zero point without much difficulty. The complete time of swing of the coil is about ten minutes, and the throws due to successive impulses of the same intensity agree with each other very closely indeed. I am much indebted to Mr. Thompson for the great skill and patience he has used in making these instruments. The apparatus was mounted for use by Mr. John Coulson, who has helped me in all the work.

When the coil of a d'Arsonval galvanometer is disturbed from its position of equilibrium and is then allowed to swing under the action of a righting moment proportional to the angular deviation from its original place, the damping effects of the resistance of the air and of the induced currents in the frame and the coil, as they move between the poles of the permanent magnet of the instrument, may usually be accounted for, with an accuracy sufficient for most practical purposes, on the assumption that the motion of the suspended system is hindered at every instant by a force-couple of moment proportional to the angular velocity. Gauss and Weber showed that this hypothesis served to explain very well the motion of the magnets which they used in their measurements at Göttingen, and they put the mathematical theory of motion resisted in this way into the form in which it appears in most treatises on Physics [1] at the present day. When,

[1] Gauss, *Resultate des Magnetischen Vereins*, 1837; W. Weber, *Resultate des Magnetischen Vereins*, 1836, 1838; *Maassbestimmungen*, vol. ii; *Math-phys. Abhandlungen der K. Sächs.* Gesellschaft, 1852; Du Bois-Reymond, *Monatsberichte der Berl. Akad.*, 1869, 1870; Chwolson, *Bulletin de St. Petersbourg*, 1881; Dorn, *Ann. der Physik*, vol. xvii, 1882; vol. xxxv, 1888; Maxwell, *Treatise on Electricity and Magnetism*, vol. ii; G. Wiedemann, *Lehre von der Elektricität*, vol. iii; Deprez et d'Arsonval, *Comptes Rendus*, vol. xciv, 1882; Riecke, *Abhandlungen der K. Gesellschaft der Wissenschaften zu Göttingen*, vol. xxx; Rachniewsky, *Lumière Élect.*, vol. xvii, 1885; see also *Lumière Élect.*, vol. xxix, 1888; vol. xxxiii, 1889; vol. xlv, 1892; Ledeboer, *Comptes Rendus*, vol. cii, 1886; Ayrton, Mather, and Sumpner, *Philosophical Magazine*, vol. xxx, 1890; vol., xlii, 1896; vol. xlvi, 1898; Classen, *Electrotechnische Zeitschrift*, vol. xvi, 1895; Sack, *Electrotechnische Zeitschrift*, vol. xvii, 1896; Des Coudres, *Zeitschrift für Electro-*

however, a system swings under strong air damping, the motion sometimes [1] departs pretty widely from the Gaussian law at the beginning, at least, and it is not always safe to apply Gauss's equations to a ballistic galvanometer which has air damping as well as electromagnetic damping until one has found out whether the ratio of successive amplitudes is fairly constant during the whole motion, as Gauss's hypothesis demands. Even in the case of one of Gauss's own magnets, the logarithmic decrement of the amplitudes increased on a certain occasion from 1168×10^{-6} to 1301×10^{-6} in 422 oscillations. It will appear in the sequel that the two long period galvanometers described in this paper follow the Gaussian law, if not exactly, still quite nearly enough to make it worth while to study their characteristics in the light of the usual theory.

The behavior of a damped ballistic galvanometer through which impulsive currents flow when the suspended system is away from its position of equilibrium and is in motion was first treated thoroughly by Dorn in a paper [2] written before d'Arsonval galvanometers were much used. In this paper Dorn studies the error introduced into observations made by Weber's methods of multiplication and of recoil, when the impulses are not properly timed. He also considers the case where the galvanometer is subjected to the action, not of a series of impulses, but of a continuous current, which lasts with given varying strength for a considerable time, and some of his equations have lately been put into other convenient forms by Diesselhorst. We shall find it desirable to derive from the beginning the special equations which we need in this paper.

The equation of motion of the coil of a d'Arsonval galvanometer, when the resisting moment is proportional to the angular velocity, is of the form

chemie, vol. iii, 1897; Barus, *Phys. Rev.*, vol. vii, 1898; Salomon, *Philosophical Magazine*, vol. xlix, 1900; Robertson, *Electrician*, vol. xlvi, 901–904; vol. xlvii, 17–20, 1901; G. Kummell, *Zeitschrift für Electrochemie*, vol. vii, 1901; Diesselhorst, *Ann. der Physik*, vol. ix, 1902; Jaeger, *Instrumentenkunde*, 1903; Stewart, *Phys. Rev.*, vol. xvi, 1903; White, *Phys. Rev.*, vol. xix, 1904; vol. xxii, 1906; vol. xxiii, 1906; Shedd, *Phys. Rev.*, vol. xix, 1904; Smith, *Phys. Rev.*, vol. xxii, 1906; A. Zeleny, *Phys. Rev.*, vol. xxiii, 1906; Wenner, *Phys. Rev.*, vol. xxii, 1906; vol. xxv, 1907.

[1] Peirce, *Proceedings of the American Academy*, vol. xliv, 1908.

[2] Dorn, *Ann. der Physik*, vol. xvii 1882, Diesselhorst, *Ann. der Physik*, vol. ix, 1902.

$$K \cdot \frac{d^2\theta}{dt^2} + 2a \cdot \frac{d\theta}{dt} + b^2\theta = 0, \tag{1}$$

where K is the moment of inertia of the suspended system about the axis of suspension. If this equation be written in the form

$$\frac{d^2\theta}{dt^2} + 2a \cdot \frac{d\theta}{dt} + \beta^2\theta = 0, \tag{2}$$

a may be called the "damping coefficient," and β^2 the "restoring coefficient." It will be convenient to represent $d\theta/dt$ by ω, $(\beta^2 - a^2)$ by ρ^2, and the complete time of swing of the coil by T.

If when $t = 0$, θ and ω have the given values θ' and ω', the general solution of (2) takes the form

$$\theta = e^{-at}\left[\theta' \cdot \cos \rho t + \frac{\omega' + a\theta'}{\rho} \cdot \sin \rho t\right], \tag{3}$$

whence

$$\omega = e^{-at}\left[\omega' \cdot \cos \rho t - \frac{a\omega' + \beta^2\theta'}{\rho} \cdot \sin \rho t\right]. \tag{4}$$

If, when the system is at rest in its position of equilibrium, an impulsive angular velocity ω_0 be given to it, and if after t_1 seconds have elapsed and the angular velocity has become ω_1, this velocity be impulsively increased by the amount ω_2, θ and ω are given during the first stage of the motion by the equations

$$\theta = \frac{\omega_0}{\rho} \cdot e^{-at} \cdot \sin \rho t, \tag{5}$$

$$\omega = e^{-at}\left[\omega_0 \cdot \cos \rho t - \frac{a\omega_0}{\rho} \cdot \sin \rho t\right], \tag{6}$$

and

$$\theta_1 = \frac{\omega_0}{\rho} \cdot e^{-at_1} \cdot \sin \rho t_1, \tag{7}$$

$$\omega_1 = e^{-at_1}\left[\omega_0 \cdot \cos \rho t_1 - \frac{a\omega_0}{\rho} \cdot \sin \rho t_1\right], \tag{8}$$

$$\rho = 2\pi/T,\ a = 2\lambda/T,\ a/\rho = \lambda/\pi,\ \beta^2 = \rho^2 + a^2.$$

If, then, for θ' and ω' in (3) and (4) we substitute θ_1 and ω_1 as given by (7) and (8), and for t in (3) and (4) put $(t - t_1)$, in order that the origin of time shall be that of (5) and (6), we shall get

$$\theta = \frac{\omega_0}{\rho} e^{-at} \cdot \sin \rho t + \frac{\omega^2}{\rho} e^{-a(t-t_1)} \sin \rho(t - t_1), \tag{9}$$

$$\omega = \omega_0 e^{-\alpha t}\left[\cos \rho t - \frac{\alpha}{\rho} \cdot \sin \rho t\right] + \omega_2 e^{-\alpha(t-t_1)}\left[\cos \rho (t - t_1) - \frac{\alpha}{\rho} \cdot \sin \rho (t - t_1)\right]. \quad (10)$$

Dorn points out that after the second impulse at $t = t_1$, the motion is the same as it would have been if there had been no such impulse, but if when $t = 0$, the values of θ and ω had been

$$-\frac{\omega_2}{\rho} \cdot e^{\alpha t_1} \cdot \sin \rho t_1, \quad (11)$$

and

$$\omega_0 + \omega_2 \cdot e^{\alpha t_1}\left[\cos \rho t_1 + \frac{\alpha}{\rho} \cdot \sin \rho t_1\right], \quad (12)$$

and shows that the formulas can easily be generalized to fit the case in which there are a number of belated impulsive changes in the angular velocity, instead of one.

In the motion represented by (3) and (4), the angular velocity vanishes at the time t' defined by the equation

$$\tan \rho t' = \frac{\rho \omega'}{\alpha \omega' + \beta^2 \theta'}, \quad (13)$$

and if the first root be used, the amplitude at the first elongation is

$$e^{-\alpha t'}\left[\theta' \cdot \cos \rho t' + \frac{\omega' + \alpha \theta'}{\rho} \cdot \sin \rho t'\right]. \quad (14)$$

For the motion defined by (5), (6), (9), and (10), therefore, the first amplitude can be found by substituting for θ' and ω' in (13) and (14) the values given by (11) and (12). The computation is, however, not very simple, and we shall do well to treat the matter graphically, using equation (9) as the basis of our work.

If we define the function $F(t)$ by the equation

$$F(t) = e^{-\alpha t} \sin \rho t \quad (15)$$

and denote the constants $\frac{\omega_0}{\rho}$, $\frac{\omega_2}{\rho}$ by p and q, (9) may be written in the form

$$\theta = p \cdot F(t) + q \cdot F(t - t_1). \quad (16)$$

For any given galvanometer with a given resistance of the coil circuit α and ρ are definite, easily determined constants, and $F(t)$ is therefore determined. For the galvanometer represented by Figure 1, Plate 1,

for instance, ρ is twice α for a coil circuit resistance of about 150 ohms. If we represent ρt by x, ρt_1 by x_1, and the ratio of α to ρ by μ, then

$$\theta = p \cdot e^{-\mu x} \sin x + q \cdot e^{-\mu(x-x_1)} \sin(x-x_1) = p \cdot f(x) + q \cdot f(x-x_1). \quad (17)$$

If then we draw the curves $y = p \cdot f(x)$, $y = q \cdot f(x)$, the ordinates of which are in the constant ratio p/q, and displace the second curve bodily to the right through the distance x_1, the sum of the ordinates of the first curve and the displaced curve will represent θ. For most

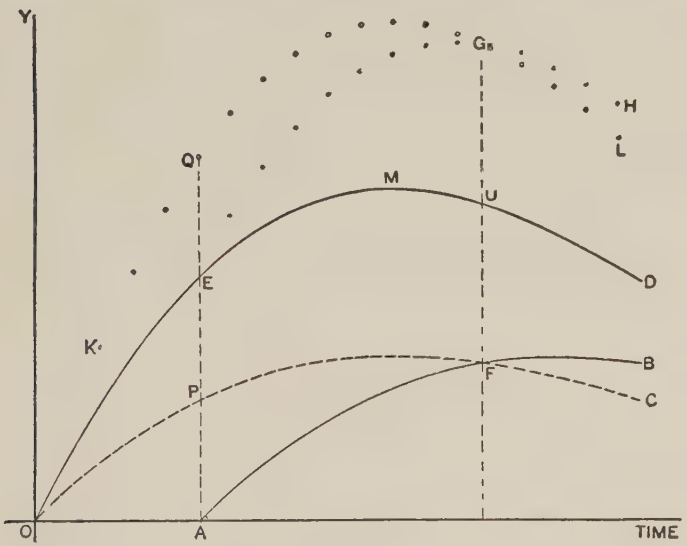


FIGURE D

purposes only the ratio (r) of q to p is important, and in plotting the curves we may make $p = 1$ and q, r.

To illustrate the process just described, let us suppose that when the galvanometer coil is at rest in its position of equilibrium, an impulsive current is sent through it, and after the coil, in response to this impulse, has had about half time enough to reach its elongation, a second impulse is given it half as strong as the first. The general form of the diagram will be much the same whether the damping be very slight or so strong that the motion is just aperiodic, but in Figure D the lines are drawn to scale for the case $\alpha/\rho = 1/2$.

OEUD is the curve $y = e^{-x/2} \cdot \sin x$, which reaches its maximum at M. OPFC is the curve $y = \frac{1}{2} \cdot e^{-x/2} \cdot \sin x$, and AFB is the last curve

moved to the right through the distance $x = \rho t_1$. The angular deviation of the coil is given as a function of ρt by the broken curve OEGH, the ordinates of which are the sums of the corresponding ordinates of OEMD and AFB. The maximum of this curve belongs to a point slightly to the left of G and measures the throw of the coil under the circumstances. If both impulses had been given to the coil when it was at rest, the deviation would have been given by the curve

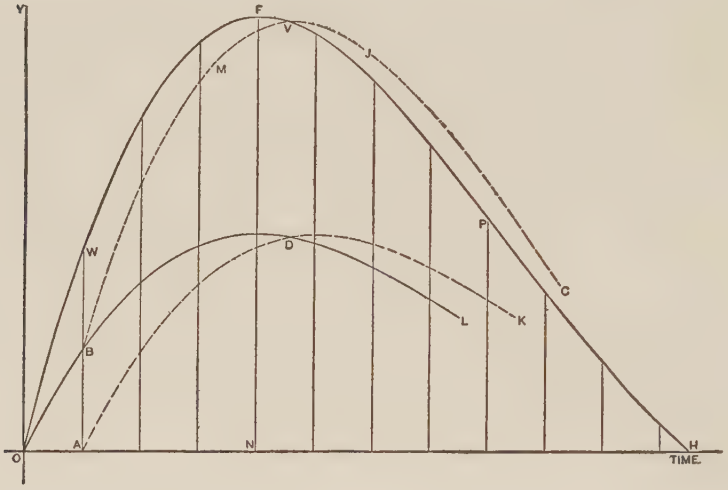


FIGURE E

OKQGL. The actual throw is about 96 per cent of the throw which would be obtained if both impulses came together at the beginning. The actual values of α and ρ are not needed, and one does not need to know the period of the coil, the actual intensities of the impulses, or anything else, besides λ and r. In this case it is easy to find out by trial in two or three minutes how great the lag OA may be if the difference of the throws is not to be greater than one half per cent, for instance.

If the secondary of an induction coil which has no iron core be connected with the coil of the galvanometer represented by Figure 1, Plate 1, and if when the current I is running steadily through the primary of the induction apparatus the primary circuit be first broken and then, after the coil has had just one quarter enough time to reach

its elongation, closed in reverse direction, the angular deviation of the coil will be given as a function of ρt by the curve OBMVJC, Figure E. The ordinates of this curve are the sums of the corresponding ordinates of OBDL and ADK. If the current in the primary circuit of the induction apparatus were suddenly reversed while the galvanometer coil was at rest in its position of equilibrium, the deviation would be given by the curve OWFPH, the ordinates of which are double those

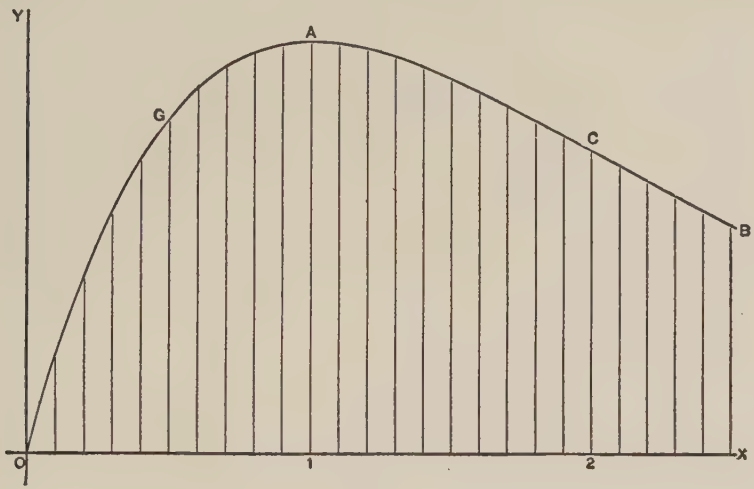


FIGURE F

of the curve OBDL. The throw with the lag OA is nearly 99 per cent of that when the current is suddenly reversed.

This graphical process is especially convenient when the allowable decrease of throw is given and one wishes to find the maximum lag which will not make the throw difference too great. If the lag is given and the throw difference is wanted, this may be found by computation, though the graphical treatment has solid advantages. It is evident that the curve $y = e^{-\lambda x} \cdot \sin x$ serves for a given galvanometer with a given coil circuit for throws of all magnitudes.

It often happens that one has to work with a galvanometer the period of which is rather too short for the purpose in hand, but it is usually possible to determine, in the manner pointed out above, a correction factor to be applied to all throws, which will make the instrument trustworthy.

When a galvanometer is critically damped $\beta^2 = \alpha^2$, $\rho = 0$, and the equation of motion is

$$\frac{d^2\theta}{dt^2} + 2\alpha \frac{d\theta}{dt} + \alpha^2\theta = 0, \tag{18}$$

and the general solution of this is

$$\theta = (A + Bt)\, e^{-\alpha t}. \tag{19}$$

If when $t = 0$, $\theta = \theta'$, and $\omega = \omega'$;

$$\theta = [\theta' + (\omega' + \alpha\theta')\, t]\, e^{-\alpha t}. \tag{20}$$

If when the coil is at rest in its position of equilibrium, an impulsive current sent through the instrument gives the coil an initial angular velocity ω_0,

$$\theta = \omega_0 \cdot t \cdot e^{-\alpha t},\ \omega = \omega_0 \cdot e^{-\alpha t}\,(1 - \alpha t), \tag{21}$$

and if after this motion has gone on until the time t_1 a second impulse increases the angular velocity by the amount ω_2, then after the second impulse

$$\theta = \omega_0 t\, e^{-\alpha t} + \omega_2\,(t - t_1)\, e^{-\alpha(t-t_1)}. \tag{22}$$

It is possible to give to this equation also a graphical treatment similar to that which we have discussed above for the case where α is less than β. If $\phi(t)$ is defined by the equation

$$\phi(t) = \alpha t\, e^{-\alpha t},$$

$$\theta = \frac{\omega_0}{\alpha} \cdot \phi(t) + \frac{\omega_2}{\alpha} \cdot \phi\,(t - t_1). \tag{23}$$

Figure F shows the form of the curve $y = xe^{-x}$.

In considering the magnitude of the throw of a damped ballistic galvanometer due to a given continuously varying current which flows through the coil for a finite time interval, we shall do well to use Dorn's results in nearly the forms into which they have been put by Dieselhorst in his important paper on the subject.

When the suspended system is at rest in its position of equilibrium, a short-lived current shall flow through the coil and shall have the intensity, I, which is a given function of the time. From the epoch $t = \tau$, I shall have the value zero. The product of the strength of the magnetic field between the poles of the permanent magnet, at the place where the coil is, and the effective area of the turns of the coil

shall be denoted by q, so that while the current is flowing, the equation of motion of the coil, for such small angles as are used in mirror instruments, has the form

$$K \cdot \frac{d^2\theta}{dt^2} + 2a \cdot \frac{d\theta}{dt} + b^2 \cdot \theta = qI \tag{24}$$

or

$$\frac{d^2\theta}{dt^2} + 2a \cdot \frac{d\theta}{dt} + \beta^2 \cdot \theta = \frac{q}{K} \cdot I = \mu \cdot I. \tag{25}$$

If, as before, m and n are the roots of the equation $x^2+2ax+\beta^2=0$, if Q_t represents the whole flux of electricity through the coil from $t = 0$ to $t = \tau$, and if M_t, N_t represent the ratios to Q_t of the integrals

$$\int_0^t I \cdot e^{-mt} \cdot dt, \qquad \int_0^t I \cdot e^{-nt} \cdot dt, \tag{26}$$

respectively, then the solution of (25) is

$$\theta = \frac{\mu}{m - n}\left[e^{-mt} \cdot \int_0^t I \cdot e^{-mt} \cdot dt - e^{-nt} \int_0^t I \cdot e^{-nt} \cdot dt\right] \tag{27}$$

$$= \frac{\mu Q_t}{m - n}\left[M_t \cdot e^{mt} - N_t \cdot e^{nt}\right]. \tag{28}$$

After the time $t = \tau$, M_t, N_t have the constant values M, N, and Q_t becomes Q, the total amount of electricity carried by the current from $t = 0$ until it ceases to flow at $t = \tau$, so that

$$\theta = \frac{\mu Q}{m - n}\left[M \cdot e^{mt} - N \cdot e^{nt}\right]. \tag{29}$$

If, as is usually true in practice, β is greater than α, ρ is positive, $m = -\alpha + \rho i$, $n = -\alpha - \rho i$, $m/(m-n) = \frac{1}{2} + \alpha i/2\rho$, $n/(n-m) = \frac{1}{2} - \alpha i/2\rho$, but the results are, of course, real.

If we determine $d\theta/dt$ from (29) and equate it to zero, we learn that at a time of elongation

$$t = \frac{1}{m - n} \cdot \log\left(\frac{nN}{mM}\right), \tag{30}$$

and this value of t substituted in (29) gives the amplitude at elongation in the form

$$A = \frac{\mu Q}{m-n}\left[\left(\frac{n}{m}\right)^{\frac{m}{m-n}} - \left(\frac{n}{m}\right)^{\frac{n}{m-n}}\right] M^{\frac{n}{n-m}} \cdot N^{\frac{m}{m-n}} \tag{31}$$

$$= C \cdot M^{\frac{n}{n-m}} \cdot N^{\frac{m}{m-n}} \tag{32}$$

where C is a function of the constants of the galvanometer and is independent of the manner in which the whole flux Q of electricity is sent through the circuit. If A_0 denote the amplitude at the first elongation when Q is sent impulsively through the coil at the origin of time,

$$\frac{A}{A_0} = M^{\frac{n}{n-m}} \cdot N^{\frac{m}{m-n}}. \tag{33}$$

If I happens to be given in analytic form as a function of t, it is possible, as Diesselhorst shows in a general case, to obtain a convergent series for A/A_0. For the purposes of this paper, however, where the form of I is shown merely by an oscillograph record, we shall find it desirable if m and n are real, to plot the curves $y = Ie^{-mt}$, $y = Ie^{-nt}$ directly from this record and then to find the values of M and N by mechanical integration.

If β is greater than α, (27) may be written

$$\theta = \frac{\mu}{\rho} e^{-\alpha t}\left[\sin \rho t \cdot \int_0^t I \cdot e^{\alpha t} \cdot \cos \rho t \cdot dt - \cos \rho t \cdot \int_0^t I \cdot e^{\alpha t} \cdot \sin \rho t \cdot dt\right]. \tag{34}$$

$$\text{If } R \cdot Q = \int_0^\tau I \cdot e^{\alpha t} \cdot \cos \rho t \cdot dt \text{ and } S \cdot Q = \int_0^\tau I \cdot e^{\alpha t} \cdot \sin \rho t \cdot dt, \tag{35}$$

the value of θ after the current has ceased is

$$\theta = \frac{\mu Q e^{-\alpha t}}{\rho}\left[R \cdot \sin \rho t - S \cos \rho t\right] \tag{36}$$

where Q, R, and S are constants.

At the first elongation,

$$\tan \rho t = \frac{\rho R + \alpha S}{\alpha R - \rho S} \tag{37}$$

or

$$\cos \rho t = \frac{\alpha R - \rho S}{\beta \sqrt{R^2 + S^2}}, \quad \sin \rho t = \frac{\rho R + \alpha S}{\beta \sqrt{R^2 + S^2}}, \tag{38}$$

and if the first root of these equations be substituted for t in (36), it appears that the first elongation is given by the expression

$$A = \frac{\mu Q\sqrt{R^2 + S^2}}{\beta} \cdot e^{-u} \tag{39}$$

where

$$u = \frac{\alpha}{\rho} \cdot \tan^{-1}\frac{\rho R + \alpha S}{\alpha R - \rho S}. \tag{40}$$

If the quantity Q of electricity had been sent impulsively through the galvanometer when the coil was at rest in the position of equilibrium, the throw would have been as (5) shows

$$A_0 = \frac{\mu Q}{\beta} \cdot e^{-v} \tag{41}$$

where $v = \frac{\alpha}{\rho} \cdot \tan^{-1}\frac{\rho}{\alpha}$.

Hence

$$\frac{A}{A_0} = \sqrt{R^2 + S^2} \cdot e^{-(u-v)} = \sqrt{R^2 + S^2} \cdot e^{-w}, \tag{42}$$

where $w = \frac{\alpha}{\rho} \cdot \tan^{-1}\frac{S}{R}$.

If $\frac{1}{2}\ Q$ were sent impulsively through the circuit at $t = 0$, and $\frac{1}{2}\ Q$ at $t = \tau$, the values of R and S to be used in (42) would be

$$R = \tfrac{1}{2}\,(1 + e^{\alpha\tau} \cdot \cos \rho\tau), \qquad S = \tfrac{1}{2}\, e^{\alpha\tau} \cdot \sin \rho\tau. \tag{43}$$

With some of the forms of short period, critically damped d'Arsonval galvanometers commonly used in American laboratories, it is difficult to reverse the current in the primary of an induction apparatus with air core by a large double throw switch so quickly as to avoid a decrease in the throw of the galvanometer coil owing to the lag in the second impulse.

If a current of constant intensity (Q/τ) flowing for the time interval τ conveys a quantity, Q, of electricity through the circuit, the values of R and S are

$$R = \frac{1}{\beta^2\tau}[e^{\alpha\tau}(\rho \cdot \sin \rho\tau + \alpha \cdot \cos \rho\tau) - \alpha] \tag{44}$$

$$S = \frac{1}{\beta^2\tau}[e^{\alpha\tau}\ (\alpha \sin \rho\tau - \rho \cdot \cos \rho\tau) + \rho] \tag{45}$$

$$\sqrt{R^2 + S^2} = \frac{1}{\beta\tau}\sqrt{e^{2\alpha\tau} - 2\, e^{\alpha\tau} \cos \rho\tau + 1}. \tag{46}$$

In the case of a critically damped instrument

$$\theta = \mu e^{-} \left[t \int_0^t I \cdot e^{\alpha t} \cdot dt - \int_0^t I \cdot t \cdot e^{\alpha t} \cdot dt \right].$$

If there were no damping, α would be zero, e^{-w} would be equal to unity, and R and S would satisfy the equations

$$RQ = \int_0^\tau I \cdot \cos \rho t \cdot dt, \qquad SQ = \int_0^\tau I \cdot \sin \rho t \cdot \mathrm{dt}.$$

The foregoing theory rests, of course, upon the assumption that the swinging system of a galvanometer meets with a resistance to its motion which may be attributed to a force couple of moment equal at any instant to the product of a fixed constant and the angular velocity which the system then has. It is evident, however, that this condition cannot be exactly fulfilled during the whole motion of the needle or coil of any instrument in which the damping soon brings the swinging system absolutely to rest. In the case of a horizontal bar magnet swinging without sensible friction about a vertical axis through its centre, the ratio of successive half amplitudes usually remains nearly constant for a large portion of the motion, though the actual value of the ratio often depends upon the atmospheric conditions, as Gauss showed. The logarithmic decrement of the oscillations of a magnetic needle swinging in a strong field under the damping action of a mica vane of the usual kind usually diminishes as the amplitudes grow smaller. The same tendency often shows itself in the case of a d'Arsonval galvanometer when the damping, either electromagnetic or atmospheric, is fairly large.

In a galvanometer of any of the common forms in which the restoring moment is due, not to the mutual action of a magnet and the external field, but to torsional forces in a spring or suspending fibre, even though the system comes to rest sensibly at its old position of equilibrium, the swings are often one-sided in a fashion best described, perhaps, with the help of an example or two.

A certain d'Arsonval galvanometer (Y) of the Ayrton and Mather type was connected in series with a rheostat of resistance R and the

coil of a small magneto-inductor. The period of the galvanometer coil was dependent of course upon the value of R: when the circuit was broken, its value was about 16.5 seconds. The same flux change in the coil of the inductor might be made over and over again at pleasure by slipping the coil in one direction or the other between two fixed stops. The resistance of the galvanometer and the inductor coil together was about 96.6 ohms. When the galvanometer coil was at rest in its position of equilibrium (scale reading 711), and the value of R was 600 ohms, the inductor coil was moved quickly from one stop to the other and a short series of turning points, 329, 886, 623, 750, 689, were observed. When the inductor coil was slipped back to its original place, the readings were 1095, 534, 799, 672, 733. Using the first set of turning points and the zero 711, the successive half amplitudes were 382, 175, 88, 39, 22, and the ratios of the successive pairs were 2.18, 1.99, 2.26, 1.77. The other set of turning points give the half amplitudes 384, 177, 88, 39, 22, and the ratios, 2.17, 2.01, 2.26, 1.77. The half sums of corresponding numbers in the two observed sets are 712, 710, 711, 711, 711, and there is no obvious bias in favor of deflections on one side of the zero point. There was no sensible "set" when the system came to rest, but during the swings there seemed to be a very slight movement of the zero point towards the side of the first excursion, at the end of which the whole angle of twist in the long gimp was only about 1°. When R was made 400 ohms, the time of swing fell from 8.6 seconds to 8.2 seconds, the throw due to the same movement of the inductor coil rose to 483, and the ratios of successive pairs of half amplitudes became 3.16, 2.68, 3.17. When the twist in the gimp per centimeter of its length is made as large as in many of the instruments in common use, the tendency here noted becomes very troublesome, and it is difficult to determine from a short set of throws corresponding to a fairly strong damping what the value of the logarithmic decrement should be.

A certain d'Arsonval galvanometer (X), of the type represented in Figure C, which was formerly in use in the Jefferson Laboratory, had a period of 149 seconds. When the coil was given a deflection corresponding to a scale reading of 14.15 cms., and was then allowed to swing, the ratios of the successive half amplitudes were 1.066, 1.061, 1.067, 1.061, 1.066, 1.060, etc.

The galvanometers (X, Y) just mentioned, unlike most of those which are usually available in a laboratory, were almost exactly symmetrical in their throws on opposite sides of the zero. In most large instruments in which the coils are wound on open metal frames, there is a slight bias, so that a given flow of electricity sent impulsively through the circuit causes a little larger throw on one side than on the other. Sometimes the bias, when the always small throw is increased by increasing the discharge, changes sign; sometimes levelling the

TABLE I

R	T	ρ	λ	α	β
3000	7.00	1.030	1.207	0.396	1.104
4000	5.95	1.056	0.699	0.234	1.082
10000	5.78	1.086	0.398	0.137	1.096
20000	5.74	1.094	0.224	0.128	1.097
Infinity	5.73	1.097	0.032	0.011	1.097

instrument will help a trifle, but usually the lack of symmetry seems to be connected with the magnetism induced in the frame or the coil by the field of the magnet.

Mr. John Coulson, who has studied in the Jefferson Laboratory the characteristics of an excellent short period d'Arsonval galvanometer of the very best make, has found a bias of about 2 per cent in favor of the throws on one side of the zero point. In this instrument there is also the same irregularity in the ratios of successive amplitudes which had been already noticed. For a given impulse, which caused a throw on one side, after which the coil oscillated with decreasing amplitude, the ratios were 2.16, 2.03, 2.15, 2.08, while the same impulse reversed in direction gave the ratios 2.09, 2.12, 2.09, 1.97. These values were persistent and could be obtained over and over, and their differences were quite large enough to disturb a person who is attempting to get an accurate value of the so-called damping coefficient for use in the differential equation.

Some of the constants of this galvanometer as determined by Mr. Coulson are given in Table I.

Such slight departures from symmetry as these seem, however, not to affect in the least the usefulness of a good d'Arsonval galvanometer in measuring quantities of electricity sent through its coil; the mean of throws on opposite sides of the zero point due to a given impulsive

discharge remains practically constant, and a good calibration might often be made to serve for a long time, though the instrument should be tested, of course, every time it is used.

In view of the fact that the motion of the coil of a d'Arsonval galvanometer usually deviates somewhat, as we have seen, from the course laid down by the Gaussian theory, we may inquire whether such equations as (14), (33), (42), based on that theory, agree with the results of observations on ordinary instruments. It may be well to say at the outset that, according to my experience, the agreement is wonderfully close.

To support this assertion I may adduce first a simple test made a long time ago upon the galvanometer X mentioned above. If we assume for λ the value 0.0611, the natural logarithm of 1.063, and for T the value 149, it appears that $\alpha = 0.00082$ and $\rho = 0.0422$. The time required for the swing out from the zero to the turning point is then $\frac{1}{\rho}\tan^{-1}\left(\frac{\rho}{\alpha}\right)$ or 36.4 seconds: the return to the zero requires 38.1 seconds. If under these circumstances a given impulse be sent through the coil, and after an interval $\tau = 10$ seconds, another equal impulse, the resulting throw should bear to that which would be caused if both impulses came together at the beginning, the ratio given by (42) when $\alpha\tau = 0.082$, and $\rho\tau = 0.422$, which corresponds to 24.18°. In this case $R = 0.9597$, $S = 0.2064$, $\sqrt{R^2 + S^2} = 0.982$, $\log e^{-w} = 9.9980$, and A/A_0 is about 0.977+. Now when a single impulse from an induction apparatus without iron was sent through the coil, and after a delay of ten seconds another equal to the first, the throw as given by a number of readings was 1144, but the reading when both came together was 1170. The ratio of these numbers is 0.978. It is easy to show by a little computation that if the delay were 5 seconds, the ratio of A to A_0 would be 0.994; but if it were 30 seconds, the ratio would be about 0.806.

Table II gives some of the results of several days' study of the characteristics of the galvanometer V. The periodic time, which was determined with the help of a chronograph, is given in round numbers, because slight differences of dampness in the air or of barometric pressure seemed to affect the period somewhat. With small values of R,

the ratio (r) of successive half amplitudes was usually somewhat variable in the manner described above, though the values were persistent. Under these circumstances the average value is given. If the instrument followed the Gaussian law exactly, the value of β should be the same throughout.

As this galvanometer was to be used in an important series of magnetic measurements during which it was necessary to determine with accuracy the change of flux in the solid core of a fairly large electromagnet when the exciting current should be reversed in direction, it

TABLE II

R	∞	500	300	200	100	50	30	0
T	132.0	132.1	132.4	132.8	133.7	136.0	137.8	143.0
r	1.23	1.34	1.40	1.46	1.68	1.94	2.19	3.12
λ	0.207	0.293	0.336	0.378	0.519	0.663	0.784	1.138
$\alpha = 2\lambda/T$	0.00314	0.00443	0.00508	0.00570	0.00776	0.00975	0.01138	0.01592
$\rho = 2\pi/T$	0.0476	0.0476	0.0474	0.0473	0.0470	0.0462	0.0456	0.0439
α/ρ	0.066	0.093	0.107	0.120	0.165	0.211	0.250	0.362
β	0.0477	0.0477	0.0477	0.0476	0.0476	0.0472	0.0470	0.0468

was desirable to study with some care the effect upon the throw due to the duration of the induced currents. If under all ordinary cases the area beneath the curve in the record of an oscillograph in series with the galvanometer is proportional to the corresponding throw of the galvanometer, one may assume that the performance of the galvanometer will continue to be satisfactory; but this test is not easy to make. It is comparatively easy, however, to give to the galvanometer coil, by aid of a large induction apparatus with air core, such a series of given impulses at given time intervals as shall give all necessary information. In fact the simple device of determining the throw due to two equal impulses separated by the interval τ for a number of different values of τ will usually serve to decide sharply whether or not the galvanometer coil follows the Gaussian law closely enough to make it possible to predict its behavior under ordinary circumstances from the equations proved above. This kind of experiment was made with Galvanometer V: an adjustable commutator, driven through a train of wheels by a motor running very steadily at just under 30 revolutions per second, served to give the impulses at the right time interval apart. A series of careful observations

showed that the throw was 1471, 1470, 1468, 1464, 1458, 1452, 1444, according as the interval between the impulses was 0, 1, 2, 4, 6, 7, or 8 seconds. At this circuit resistance, $T = 139$, $\rho = 0.0450$, $\alpha = 0.0125$, and if we assume the interval to be 8 seconds, $\alpha\tau = 0.1$, and $\rho\tau = 0.360$, which corresponds to 20.63°. According to (43) under these conditions, $R = 1.017$, $S = 0.195$, $\sqrt{R^2 + S^2} = 1.035$, $\tan^{-1}(S/R) = 0.1891$, and $A/A_0 = 0.982$. That is, the throw when the second impulse follows the first at the interval of eight seconds should theoretically be only 982 thousandths of the throw due to the two impulses coming together. The results of experiment give 1444/1417 or 0.982. This exact coincidence is, of course, a matter of chance.

When the interval is 4 seconds, $\alpha\tau = 0.05$, $\rho\tau = 0.180$, and $A/A_0 = 0.995$; here again the agreement with observation is exact for 1464/1471 = 0.995. For an interval of 6 seconds, theory gives for A/A_0 the value 0.992+ and experiment, 0.992−, so that the experimental results, obtained long before any computations were made, point to a complete agreement, within the limits of observation, with theory.

With this damping, corresponding to a value for R of about 25 ohms, the time required for the coil to reach its elongation from the zero point is about 28.9 seconds; the return takes 40.6 seconds. When R is 500, the time from the zero is 32.9 seconds, and the time back is 33.1 seconds.

When the circuit of the exciting coil of a large electromagnet is suddenly broken, the induced current in a test coil wound around the core rises very quickly to a maximum value and then falls away gradually: indeed the form of the current is usually much like that in the secondary circuit of an induction coil with air core when the primary current is suddenly interrupted. Such a current is shown by curve P of Figure G,[1] which is drawn for the case $M = L/2$ when the self-inductances of the two circuits are equal. If, after the current in the exciting coil of an electromagnet has been running steadily, its circuit be broken and after a short interval closed again, the induced current in the test coil will be very different according to the direction of the current in the main circuit. If the new direction is the same as that of the current before the break, the new current is called "direct," but if the new direction is opposed to the old, the new current is said to be "re-

[1] Identical with Fig. 2 on p. 103.

versed." The curves M, N in Figure H,[1] which are reproduced to scale from the records of an oscillograph, show the manners of growth of reversed and direct currents, respectively, in the exciting circuit of a certain electromagnet; and the boundaries of the shaded portions of the diagram show the forms of the induced currents. The shaded areas give the whole transfer of electricity in the induced currents in the two cases. Besides the exciting coil, this magnet had another similar coil wound about the core. Curves V and W show the growth of reversed and direct currents in the exciting circuit when the last-named coil was closed on itself, and the currents induced in it hindered the establishment of the main current. The scale of the oscillograph in the secondary circuit was different from that used before, but the general shape of the induced current is shown by the boundary of the shaded area v. Curves C and F of Figure I [1] show the forms of induced currents in the testing coil in the case of a very large magnet the cross section of the solid core of which had an area of about 500 square centimeters. A and D show the corresponding currents in the main circuit: in the first case the generator was a battery of 40 storage cells, and a considerable amount of extra resistance was used in the circuit; in the second case the same final current was caused by a battery of 10 cells, and very little extra resistance was needed. This particular engraving, which was made by the "Wax Process," does not reproduce the original exactly, for the upper portions of A and D are here too nearly horizontal.

A very uncommon form of secondary current is shown in Figure J.[1] Curve 1 represents the form of the main current of a very large electromagnet with massive core. At the axis of a portion of the core was a longitudinal hole about an inch in diameter, and in this hole was inserted an iron rod around which a layer of insulated wire was wound to serve as a test coil. Curve 2 shows the form of the induced current in this coil when the main circuit was closed; the dotted curve gives the form of the induced current when the main circuit was suddenly broken. The crest of the curve 2 does not come until fourteen seconds after the main current starts.

Figure K [1] shows the manner of growth of a current of final intensity

[1] Figs. H, I, J and K are identical with Figs. 48, 20, 21 and 30 on pp. 153, 81, 83 and 135 respectively.

2.3 amperes, under a voltage of perhaps 60, in a coil of 1388 turns about the core of the magnet depicted in Figure A. The curve OTJN is a copy of the record of an oscillograph in the circuit when the electromotive force was suddenly applied at $t = 0$. The area between this curve and its asymptote up to any value of the time represents the whole change of the flux of magnetic induction through the coil, and the difference between the ordinate of the asymptote and that of the curve is proportional to the instantaneous rate of change of this flux, and, therefore, to the induced electromotive force in a test loop passed around the core. The general form of the induced current in such a secondary circuit might be seen by looking at the curve just mentioned upside down and through the paper. In this case the induced current would practically come to an end in about five and one half seconds. The line OZRXUPQN shows the growth of the main current when there was an extra non-inductively wound resistance in the circuit which was suddenly shunted out after about five and one half seconds. Here, again, the general shape of the induced current in the secondary circuit might be seen by looking at this line upside down, from behind. The intensity of the induced current was inappreciable after about eight seconds.

Figure L[1] shows the general shape of the induced currents in the circuit of a test coil of a few turns wound on the core of an electromagnet when the current in the exciting circuit is made to grow by shunting out a part of the resistance of this circuit by steps. If the currents, up to the time OQ were sent through the coil of a long period ballistic galvanometer, the resulting throw would not fall so much below the throw due to the whole quantity of electricity carried by the currents, sent instantaneously through the galvanometer at the origin of time, as would the throw due to a steady current lasting for the time OQ and carrying the same total amount.

The examples already given will serve well enough to show what is required of a galvanometer which shall measure accurately the whole quantity of electricity which flows in the test coil. Of course, the induced current may last with an extremely feeble intensity for a long time, but in any practical case it is easy to set a limit of time after which no sensible flow will occur.

[1] Identical with Fig. 9 on p. 113.

If A_0 is the throw which would be caused by an instantaneous discharge of Q units of electricity through a galvanometer at the beginning of motion, A' the throw caused by an instantaneous discharge of $\frac{1}{2}\ Q$ units at the beginning and another discharge of $\frac{1}{2}\ Q$ units seconds later, and A'' the throw due to a steady current of Q/τ units intensity lasting from $t = 0$ to $t = \tau$, then A' is less than A'', and this in turn is less than A. Occasionally one encounters an induction current which has a form much like that indicated in Figure N [1] by the curve KLG, and we shall find it interesting to determine the ratio A''/A for one or two practical cases. It is well to notice that the second member of (42) depends only upon the ratios $\lambda = \alpha/\rho$ and $\delta = \tau/T$, and not at all upon the other constants of the instrument; for if we write $z = \rho t$ and $I = f(t) = \phi(z)$, we shall find that

$$R = \frac{\int_0^{2\pi\delta} \phi(z) \cdot e^{\lambda z} \cdot \cos z \cdot dz}{\int_0^{2\pi\delta} \phi(z) \cdot dz}, \qquad S = \frac{\int_0^{2\pi\delta} \phi(z) \cdot e^{\lambda z} \cdot \sin z \cdot dz}{\int_0^{2\pi\delta} \phi(z) \cdot dz}, \tag{47}$$

and these expressions involved λ and δ but are independent of the sensitiveness of the galvanometer and of its time of swing.

It is possible to show from equations (44) and (45), after some computation, that for the case of the galvanometer V, for which we may take $\alpha = 0.0125$, $\rho = 0.0450$; $A''/A_0 = 0.994$, or 0.998, according as τ is 8 seconds or 4 seconds. It is well to recall the fact mentioned above, that $A'/A = 0.982$ or 0.995, according as $\tau = 8$ seconds or 4 seconds.

Perhaps most of the induction currents which one meets in making magnetic measurements have forms similar to those of the curves S or P in Figure G, and it is worth while to compute the value of the ratio A/A_0 on the supposition that the current flows from $t = 0$ to $t = \tau$ with the intensity $I = k(\tau - t)$ where it is clear that $k = 2Q/\tau^2$.

$$\text{Since } \int x \cdot e^{\lambda x} \cdot \sin x \cdot dx = \frac{e^{\lambda x}}{(1+\lambda^2)^2}\left[\lambda \cdot \sin x - \cos x)\ (\lambda^2 x + x - \lambda) + (\sin x + \lambda \cdot \cos x)\right], \tag{48}$$

$$\text{and } \int x \cdot e^{\lambda x} \cdot \cos x \cdot dx = \frac{e^{\lambda x}}{(1+\lambda^2)^2}\left[(\sin x + \lambda \cdot \cos x)\ (\lambda^2 x + x - \lambda) - (\lambda \cdot \sin x - \cos x)\right], \tag{49}$$

[1] Identical with Fig. 55 on p. 160.

it is not difficult to prove that when $I = k(\tau - t)$,

$$R = \frac{2}{\beta^4 \cdot \tau^2}[\alpha \cdot e^{\alpha\tau}(\rho \cdot \sin \rho\tau + \alpha \cdot \cos \rho\tau) + \rho \cdot e^{\alpha\tau}(\alpha \cdot \sin \rho\tau - \rho \cdot \cos \rho\tau) + \rho^2 - \alpha^2 - \alpha\beta^2\tau], \quad (50)$$

$$S = \frac{2}{\beta^4 \cdot \tau^2}[\alpha \cdot e^{\alpha\tau}(\alpha \cdot \sin \rho\tau - \rho \cdot \cos \rho\tau) - \rho \cdot e^{\alpha\tau}(\rho \cdot \sin \rho\tau + \alpha \cdot \cos \rho\tau) + \beta^2\rho\tau + 2\alpha\rho]. \quad (51)$$

These formulas are not very well adapted for easy computation, and in many practical cases in which the quantities in the brackets are

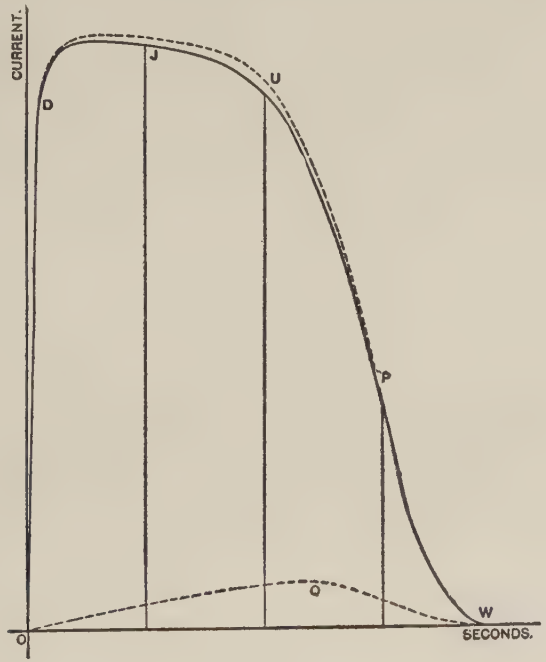


FIGURE O

very small and the coefficient $2/\beta^4\tau^2$ very large it is desirable to use five or six place logarithms in the work. As an illustration of the use of these equations we may consider the instance of the galvanometer V through which a current of the form $I = k(\tau - t)$ shall flow for 8 seconds. Here $\alpha = 0.0125$, $\rho = 0.0450$, $\beta^2 = 0.0021812$, $2/\beta^4\tau^2 = 6568.39$, $R = 1.04723$, $S = 0.12545$, and $A/A_0 = 0.9974$. The throw

due to this current is the same within about one quarter of one per cent as if the whole amount of electricity conveyed by the current had been sent instantaneously through the coil at the time $t = 0$. For a galvanometer of the same period with practically no damping the value of A/A_0 under the circumstance just mentioned would be about 0.9964. A current of the form $I = k\,(\tau - t)$ and lasting for 34 seconds would, in the case of the galvanometer W, give a throw within about one third of one per cent the same as an impulsive discharge of the same total amount would cause if sent through the coil at the origin of the motion.

For a current of the general shape of S (Figure G) regarded as stopping at the time $t = \tau$, the ratio of A/A_0 would be much more nearly unity than for a current of the form $I = k\,(\tau - t)$.

If as in the case of an induction coil without iron, when the primary circuit is suddenly broken, I is of the form $I_0 \cdot e^{-kt}$, and if we write $g = \alpha - k$,

$$RQ = \frac{I_0}{g^2 + \rho^2}\left[e^{g\tau}\,(\rho \cdot \sin \rho\tau + g \cdot \cos \rho\tau) - g\right], \tag{52}$$

$$SQ = \frac{I_0}{g^2 + \rho^2}\left[e^{g\tau}\,(g \cdot \sin \rho\tau - \rho \cdot \cos \rho\tau) - \rho\right], \tag{53}$$

$$Q = \frac{I_0}{k}\,(1 - e^{-k\tau}). \tag{54}$$

If $g = -\frac{1}{2}$, $\alpha = 0.0125$, and $\rho = 0.0450$; the value of A/A_0 will be 0.989, if the current flows until the needle reaches its elongation, say for 29 seconds.

When the shape of an induced current which is to pass through a ballistic galvanometer of long period is not analytically simple, it is always possible to determine by mechanical integration, with sufficient accuracy, the ratio of the throw caused by the current to the throw which the same total quantity of electricity sent instantaneously through the instrument would give. As an example, we may consider the form of current represented by the curve ODJPW of Figure O, which is a fairly close copy of an oscillogram. If we assume that the duration of the current is to be 4 seconds and that galvanometer V is to be used, so damped that

$$\alpha = 0.0125, \qquad \rho = 0.0450,$$

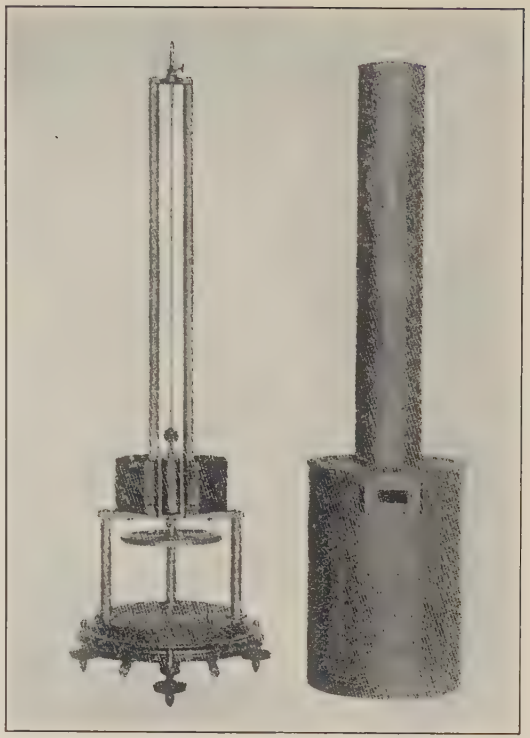


Figure 1

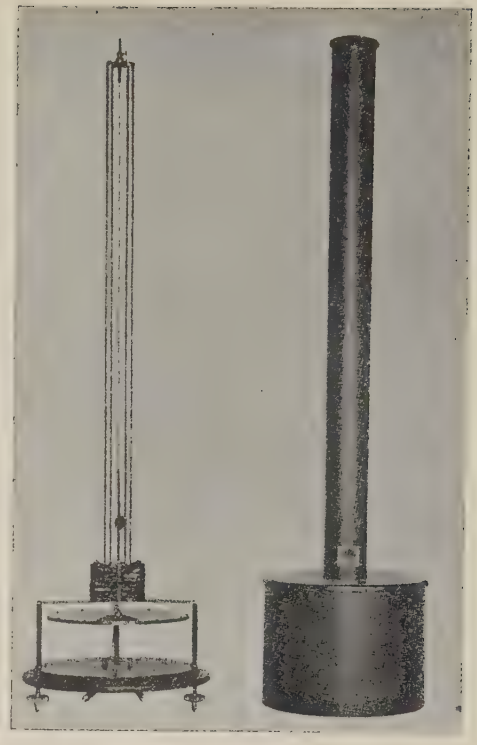


Figure 2

it is easy to measure a number of ordinates of the current curve, multiply each by the corresponding values of $e^{at} \cdot \cos \rho t$, $e^{at} \cdot \sin \rho t$, and thus compute the ordinates of the curves OUPW and OQW. The areas under these curves obtained by a good planimeter represent RQ and SQ of (35) and (42), and the area under the current curve gives Q on the same scale. An actual trial would show that A falls below A_0 by about one seventh of one per cent. If the galvanometer W were used, it would be quite impossible to detect the difference between A_0 and A, even if the duration of the current, of the form shown, were as much as 16 seconds.

The galvanometers V and W are to be used in making determinations by the "Isthmus Method" of the ultimate values of the intensity of magnetization in a large number of specimens of magnetic metals, in cases where it is necessary to reverse the direction of the exciting currents. When a rather small yoke which weighs about 300 kilograms is used under a fairly high voltage, V works very well: the whole duration of the induced current is practically less than 5 seconds, and the intensity falls off rapidly after the first, so that the difference between A and A_0 is wholly inappreciable. For very high values of the induction a solid yoke of the form shown in Figure B is to be employed. In this case the smallest cross section of the core has an area of 450 square centimeters, and it is not possible sensibly to reverse an excitation of say one hundred and fifty thousand ampere turns about this core in less than about 30 seconds under any practicable voltage. Of course the process is not completed even in this time, but the amount of electricity carried by the induced current after 30 seconds can be made relatively very small. Indeed for the shape of current practically encountered with this apparatus, the duration of the flow might be 60 seconds without causing a decrease of more than a fraction of one per cent in the throw of the galvanometer W.

I wish to express my obligation to the Trustees of the Bache Fund of the National Academy of Sciences for the loan of apparatus used in studying for this paper some of the induction current diagrams.

XII

ON THE PERMEABILITIES AND THE RELUCTIVITIES, FOR VERY WIDE RANGES OF EXCITATION, OF NORMAL SPECIMENS OF COMPRESSED STEEL, BESSEMER STEEL AND NORWAY IRON RODS[1]

When a rod or a closed frame of iron becomes magnetized under the action of a steady electric current in an exciting coil of insulated wire wound about it, the flux of magnetic induction (B) through any cross section of the iron can be easily determined with the aid of a small testing coil, but it is often very difficult to tell just what the value of the exciting magnetic field (H) is at any given point within the metal. In a few familiar cases, however, the difficulty disappears.

If, for instance, a homogeneous round rod of soft iron the length of which is, say, five hundred times the diameter, be placed in a solenoid of narrow bore, somewhat longer than the rod and uniformly wound with n turns of insulated wire per centimeter of its length, and if a steady current (C) be sent through the solenoid, the demagnetizing effects of the ends of the rod become inappreciable near the center, O, and we may assume without sensible error that the value of H at every point of a cross section in the neighborhood of O is equal to the value ($4\pi nC/10$) of H just without the rod at O.

In the case of a soft iron toroid, uniformly wound with N turns (in all) of insulated wire, the value of H is, of course, not the same at every point of a meridional section of the metal, but if the material is perfectly homogeneous, there is practically no leakage of induction into the air. Under these circumstances, the magnetomotive force is the same ($4\pi NC/10$) around all closed non-evanescible paths in the iron, and the value of H is inversely proportional to the distance from the axis of revolution of the toroid. In practice, such a toroid must be turned out of a solid piece of the metal, for it is almost impossible to make a ring out of iron rod, bent and welded together at the ends,

[1] *American Journal of Science*, 4th Series, vol. xxvii, no. 160, April, 1909.

which shall be sufficiently homogeneous to prevent serious leakage of lines of induction, and H is often very different in such a ring at different points of a circumference in the iron coaxial with the ring. If a closed magnetic circuit be made by putting a piece of soft iron rod lengthwise between the jaws of a massive yoke of any of the common forms, there is usually such an amount of leakage through the surface of the rod that the induction flux has very different values at different sections of it. However carefully the joints are made between the jaws and the specimen, it is very difficult to compute, from previous determinations of the magnetic properties of the rod and the yoke, what portion of the whole magnetomotive force of the circuit is used in the rod; indeed the fraction is very different for different excitations, and for the same soft yoke may depend very much upon the hardness of the piece to be studied. For comparatively low excitations up to say $H = 100$, a slender yoke may be used so that the cross section of the magnetic circuit is not very different at different places, and the exciting coils can then be so arranged on the yoke and the specimen, if the joints are well made and the whole circuit is magnetically fairly homogeneous, that the induction flux is nearly the same throughout. It is understood that this procedure [1] has been brought to great perfection in the National Bureau of Standards at Washington. For excitations of $H = 2000$ or more, however, the arrangement seems hard to manage.

A magnetic field in air near a magnet is both solenoidal and lamellar, and if in any portion of such a field the lines are sensibly straight and parallel, we may infer that in that region the field is practically uniform. If within a piece of perfectly soft iron the magnetization vector (I) has everywhere the direction of the exciting magnetic force, H, and an intensity expressible in terms of H alone, the induction in the metal is a solenoidal vector which has the same lines as the lamellar vector H. If throughout any region within such a piece of iron the lines of force are straight and parallel, the magnetic force and the induction are both uniform in that region. At the surface of separation between iron and air, the tangential components of the force are continuous, but the normal component of the force is generally discontinuous; if the lines of force in air just outside the surface are

[1] Burrows, *Proc. American Soc. for Testing Materials*, vol. viii, 1908.

parallel to it, the lines of induction in the metal at this surface are parallel to the surface. If, then, the magnetic field around a slender rod of iron magnetized lengthwise, has, near the surface of the rod, lines parallel to the rod's axis, we may inquire whether the lines of force and of induction within the rod are not in this neighborhood all parallel to the axis, so that the value of H throughout a cross section of the rod is the same as the value just outside the iron.

The assumption that the value of H in the air just about a slender neck of iron of proper form held between the jaws of a highly excited electromagnet is the same as the value in the neck itself, lies at the foundation of the "Isthmus Method" of determining permeabilities under very high excitations introduced by Ewing and Low.[1] According to my somewhat extended experience the so-called "maximum value of I" may be determined by the Isthmus Method to within 1 or 2 per cent of the truth if the poles and the test-piece are of the proper shape and are properly connected, and if the jaws as well as the isthmus are fairly soft; but these conditions are not always easy of attainment, and if one assumes them to be satisfied without investigating each case by itself, one may be led into grievous error. The field about a hard steel isthmus between soft jaws is usually far from uniform, and for some specimens of hard steel which I have studied I have not yet succeeded in obtaining by the Isthmus Method trustworthy determinations of I_∞. In some instances the values of H in the air were manifestly smaller or larger than in the isthmus, and sometimes they were smaller for one excitation and larger, for the same isthmus, for another excitation. Nevertheless the method is, of course, a most valuable one.

This paper describes a long series of determinations of the permeabilities of normal brands of compressed steel, Bessemer steel, and Norway iron in the form of half-inch rods, over a wide range of excitation, and it considers especially a method of measurement in a massive yoke in the interesting region from $H = 400$ to $H = 2500$ which lies above the limits of most permeameter observations and below those of the Isthmus Method. The work was undertaken in

[1] Ewing, *Magnetic Induction in Iron and other Metals;* Ewing and Low, *Philosophical Trans.*, vol. clxxx, 1889.

order to determine the magnetic behavior, in the region just mentioned, of an important specimen of soft iron of which only a single short piece was available, and it was necessary to test the trustworthiness of the method to be used by applying it to some soft metals which could be obtained in large pieces, and the permeabilities of which could be otherwise found, at least approximately.

If the rod to be experimented upon can be kept cool artificially, it is not very difficult to determine accurately in a very long solenoid, the permeability of a soft, homogeneous rod four or five hundred diameters long, for excitations up to $H = 400$. Here the value of I is probably between 95 per cent and 97 per cent of the final value (I_∞), which can be found by the use of the Isthmus Method. If, then, any other method of measuring permeability be used on the specimen between $H = 300$ and $H = 2000$ and if this method yields the proper values of I at both ends of the interval, it is comparatively easy to judge, from a graphical representation of all the observations, whether the short interval corresponding to from 3 to 5 per cent of I_∞ is properly bridged. The principal difficulty with this procedure is that several isthmus specimens of a metal and several testing coils must be used before one can be satisfied that the resulting value of I_∞ is correct to within 1 per cent; for, as Ewing has shown, the separate results of a series of determinations of I_∞ by the Isthmus Method may differ from the mean on either side by as much as 4 per cent. As a matter of fact, in all the materials I have tested the final value of I obtained by the Isthmus Method does not differ by so much as 1 per cent from the final value as obtained by the other method I have used. This second method, however, gives in any case a series of determinations of I_∞ which do not as a rule range over so much as 1/3 per cent of the mean, while the Isthmus Method in my hands is much less satisfactory in this respect.

The Bessemer and the compressed steel were procured in specially long pieces from which lengths of about 450 centimeters were cut, and these, under the severest test which I could conveniently apply, seemed to be practically uniform throughout. The Norway iron, on the contrary, was not everywhere of the same temper and could not be used satisfactorily until it had been carefully annealed by Mr. George W. Thompson, the mechanician of the Jefferson Laboratory,

who has had much experience in work of this kind. Upon each of these rods at its center a test coil of fine wire was wound by Mr. John Coulson, who has made all the test coils and has helped me in every part of the work, and then the coil and its leads were carefully covered by pieces of rubber and rubber tape to make the whole waterproof. The rod thus prepared was placed inside a horizontal solenoid nearly 500 centimeters long, placed perpendicular to the meridian. This solenoid was uniformly wound with 20,904 turns of well-insulated wire and a stream of tap water could be kept running through the bore around the rod to prevent any appreciable rise of temperature. The rod was demagnetized in situ by means of a long series of currents in the solenoid, alternating in direction and gradually decreasing in intensity, and then a curve of ascending reversals was obtained in the usual manner. It is to be noticed that the demagnetizing process does not succeed unless the rod is practically homogeneous throughout its length.

The ballistic galvanometer employed in this work has been already described [1] in a previous paper, and it is only necessary to say here that its period was so long that no detectable error was introduced into its indications by the fact that four or five seconds were necessary to make the magnetic changes corresponding to a reversal of current in the exciting coil of the yoke. For the largest currents a battery of about 120 large storage cells was used; a battery of 30 or 40 cells furnished the weaker currents.

When it is necessary to create inside an iron rod a fairly uniform magnetic field of intensity much greater than, say 500, some kind of yoke [2] is almost indispensable, and many kinds of yoke-permeameters

[1] Peirce, *Proc. Amer. Acad. of Arts and Sciences*, vol. xliv, 1908.

[2] H. E. J. G. DuBois, *Phil. Mag.*, 1890, "The Magnetic Circuit in Theory and Practice"; Shuddemagen, *Proceedings Amer. Acad. of Arts and Sciences*, vol. xliii, 1907; J. Hopkinson, *Phil. Trans.*, vol. clxxvi, 1885; Drysdale, *Electrician*, vol. xxviii, 1901; Thornton, *Electrician*, vol. xlix, 1902; Ewing, *Electrician*, vol. xxxvii, 1896; Schmidt, *Ann. der Physik*, vol. liv, 1895; Chattock, *Electrician*, vol. xxxvii, 1896; Lamb and Walker, *Electrician*, vol. xlvii, 1901; Baily, *Electrician*, vol. xlviii, 1901; Köpsel and Robinson, *Electrische Zeitschrift*, vol. xv, 1894; Kath, *Electrische Zeitschrift*, vol. xix, 1898; Kennelly, *Electrische Zeitschrift*, vol. xiv, 1893; Blondel, *Comptes Rendus*, vol. cxxvii, 1898; Stoletow, *Ann. der Physik*, vol. cxlvi, 1872; Rowland, *Phil. Mag.*, vol. xlvi, 1878; Ewing, *Magnetic Induction*, 1892; Behn-Eschenberg, *Electrische Zeitschrift*, vol. xiv, 1893; Kapp, *Electrical Engineer*, vol. xxiii, 1894.

are now used successfully in studying the magnetic properties of short rods at commercial excitations. For very high excitations, at which the air is nearly as permeable as the metal, the leakage becomes very troublesome and depends upon matters which cannot be easily controlled. I hoped, notwithstanding this fact, to be able to calibrate the yoke we had (figure 3) by means of standard pieces and thus make it available for studying short pieces of iron at excitations of about $H = 1000$, but Mr. John Coulson and I worked for a long time on this problem without finding any satisfactory solution. We found it possible, however, to determine the length of a half-inch rod of iron which, mounted between the jaws of this particular yoke, would cause the magnetic field in the air just without the iron near the middle of the specimen to be practically straight and parallel to the rod for a considerable distance. A piece about 15 centimeters long, accurately fitted for about 3.5 centimeters at each end into the taper holes in the jaws and leaving 80 millimeters of the rod free, satisfied the conditions, and this length might be slightly varied, but a much shorter rod violated the conditions in one direction (the determination of μ being too large) and a longer rod in the other. If, then, a testing coil (K) of very fine wire were wound in a single layer over a centimeter or two of the middle of a rod of the standard length, and a second, similar coil (L) of radius about two millimeters greater than that of K, upon an extremely thin non-magnetic spool slipped over K, it was to be expected that a knowledge of the whole amounts of the currents induced in K and L when the exciting current of the yoke was reversed would determine, as in the Isthmus Method, corresponding values of H and B. This assumption was fully justified by experiment. The radii of the coils and the diameter of the wire were measured with the help of a Zeiss comparator (No. 3196). The coil L was wound upon a very dry piece of boxwood, and was carefully baked in shellac. Paraffine wax is inadmissible as an insulator on the wire because of its magnetic properties, and for the same reason a vulcanite spool cannot be used for L.

The moment of inertia of the suspended coil of the long period ballistic galvanometer used for the work was so great that at low excitations the instrument was rather insensitive when used to measure the difference of the fluxes through K and L, and it became neces-

sary to make the scale distance nearly four meters. Under these circumstances it was practically impossible with a very large reading telescope of the very best make (Clark's) to get a bright image of the scale sufficiently magnified to render ballistic determinations easy, so

FIGURE 1

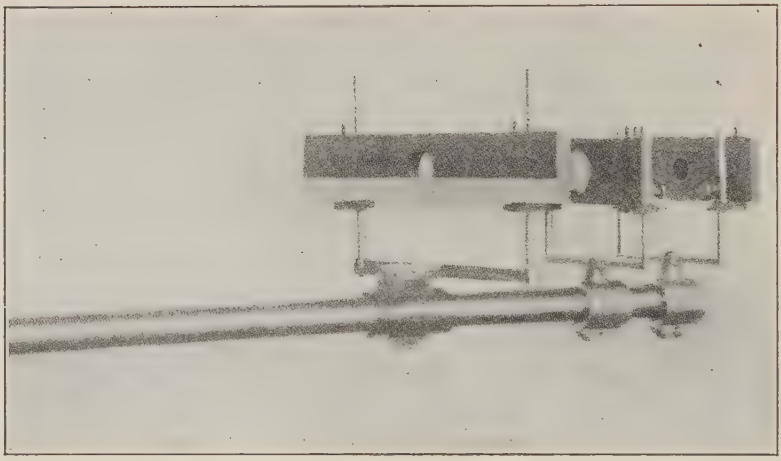


FIGURE 2

we had recourse to the simple device [1] represented by figures 1 and 2, and this has given great satisfaction. In front of the reasonably plane mirror (M) of the galvanometer, instead of the usual cover glass, was placed a convex spectacle lens (A) of about four meters focal length. At a distance in front of A equal to its focal length was a horizontal scale (S) mounted in the usual manner on a thin strip of wood at least twice as wide as the scale itself. Through the middle of

[1] Peirce, *Proc. Amer. Acad. of Arts and Sciences*, vol. xlii, 1906.

this strip above the scale was bored a round hole (H) rather more than 20 millimeters in diameter, and just behind the scale a fine vertical wire or silk fibre (W) was stretched across the opening to serve as a cross hair. Behind H, and at a distance suited to its focal length and to the eye of the observer, was placed another spectacle lens (B) to serve as an eyepiece. This had a focal length of about 20 centimeters. A peephole (P) on the common axis of H, A, and B, so placed that B's aperture appeared wholly filled with a large clear uncolored

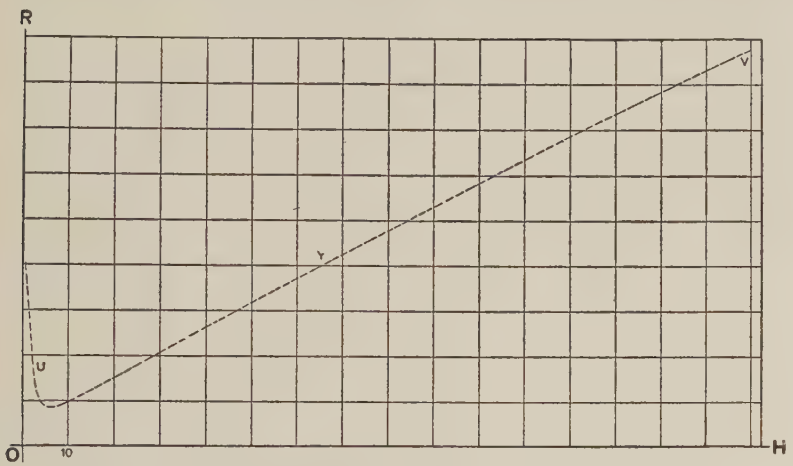


FIGURE 5. — Curve showing the reluctivity of the soft Bessemer steel rod for values of H between 3 and 160.

image of the scale with the crosswire running vertically across it, completed the arrangement. If different persons used the device the position of B had to be changed to suit the eyes of the observer, but the position of H was fixed. In setting up apparatus of this sort, one should remember that if the distance between B and P is not properly chosen, only a small round portion of the image of the scale will be visible, not nearly large enough to fill the aperture of B.

The electromagnetic yoke used in the experiments described here is represented in figure 3.[1] The three exciting coils are wound upon brass spools and these are mounted upon pieces of soft steel shafting carefully turned to fit the holes in the heavy castings which complete the frame. The coils have together 2956 turns and the wire of which

[2] Identical with Fig. 1 on p. 211.

they are made is large enough to carry a current of 50 amperes for a few minutes at a time without undue heating. The whole yoke weighs about 300 kilograms. The apparatus is supplied with jaws of a number of different forms, but for the results recorded below conical jaws of the form shown in figure 4 [1] with taper holes to receive the tapered ends of the specimen to be tested were used. All the joints were made by Mr. Thompson and seemed to be mechanically perfect. Before

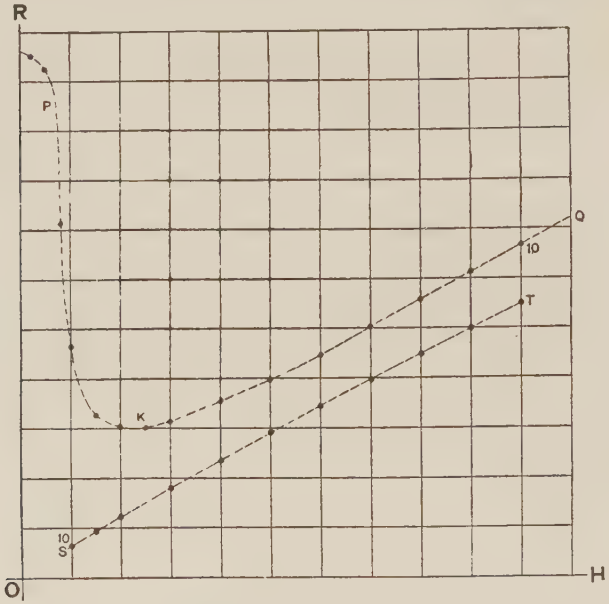


FIGURE 6. — The ordinates of the upper curve show the values of the reluctivity of the annealed Norway iron for values of H between 0 and 10. The ordinates of the lower curve represent, on a scale one-tenth as large, the reluctivities of this iron for excitations between $H = 10$ and $H = 100$.

each piece was tested the jaws were driven together with the test-piece between, and then the whole was clamped.

For very low excitations the field about the test-piece did not seem to be quite uniform, but this difficulty disappeared when the exciting current became as strong as one ampere.

The intensities of the exciting currents were measured with the

[1] Identical with Fig. 2 on p. 212.

help of a set of five Weston amperemeters of different ranges, all but one of which could be shunted at pleasure out of the circuit. The currents passed through a rack of three rheostats specially made for the purpose by the Simplex Electric Company: these had a range of about 8000 ohms.

The ballistic galvanometer was calibrated at short intervals by means of a coil (X) of 13.6 ohms resistance always in its circuit. This coil, X, consisted of 534 turns of fine insulated wire wound in a single

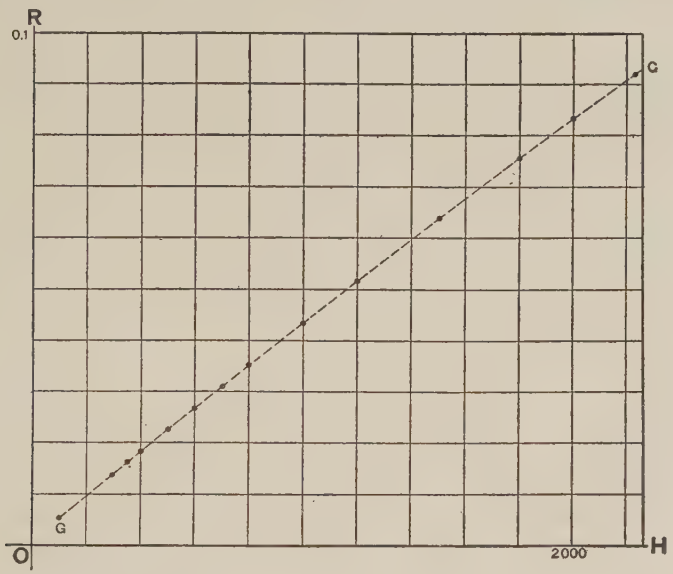


FIGURE 7. — This curve shows the reluctivity of the annealed Norway iron for values of H between 100 and 2200.

layer upon a very accurately cylindrical wooden core which was hung within a uniformly wound vertical solenoid 176.2 centimeters long consisting of 5526 turns. The effective area of the turns of X was 22,720 square centimeters and a current of one ampere sent through the solenoid caused a flux of 895,400 maxwells through X.

The remarkable solenoid, nearly five meters long, in which the permeabilities were determined for values of H below 410, was constructed by Mr. Thompson and his assistants in a lathe the bed of which had been temporarily lengthened. The core was a thick-walled solid-drawn brass tube of about one inch inside diameter made by the

American Tube Works. The solenoid was mounted on a long oak truss which kept it from buckling.

Every specimen was tested in the solenoid several times over to make sure that the demagnetizing process was so effective that at low excitations, where differences are often apt to appear, the same results were obtained at every trial. Until the Norway iron had been specially annealed, it was impossible so to demagnetize the rod that for low currents the induction should always be exactly the same. As Shuddemagen [1] has shown, the "end correction" for a round straight rod is a function of the absolute dimensions of the rod and not merely of the ratio of the length to the diameter, but the correction in the case of a half-inch rod of the length used in the work here described, is very small.

No other explanations seem necessary to make the following tables intelligible.

TABLE I

BESSEMER HALF-INCH ROD IN LONG SOLENOID

H	B	H	B
0	0		
0.5	115	10.5	10650
1.0	280	11.	10890
1.5	465	15.	12400
2.0	730	20.	13360
2.5	1180	30.	14480
3.0	1935	40.	15200
3.5	2800	50.	15720
4.0	3760	60.	16120
4.5	4830	70.	16460
5.0	5700	80.	16750
5.5	6410	90.	17000
6.0	7060	100.	17220
6.5	7650	120.	17640
7.0	8130	160.	18240
7.5	8600	200.	18800
8.0	9040	240.	19200
8.5	9420	270.	19620
9.0	9760	300.	19800
9.5	10070	350.	20240
10.0	10390	400.	20660

Numerous observations were made for values of H lying between 350 and 1600 and these were used to draw a curve from which some

[1] Shuddemagen, *Proc. Amer. Acad of Arts and Sciences*, vol. xliii, 1907; DuBois, *The Magnetic Circuit*, § 31.

TABLE II

HALF-INCH BESSEMER ROD IN MASSIVE YOKE

(*Free length about 80 mm.*)

H	*B*	*I*
400	20660	1613
500	21250	1652
600	21660	1676
707	21900	1686
800	22020	1688
1000	22230	1689
1208	22470	1692
1500	22770	1692
2090	23550	1692
2590	23860	1693

of the numbers given above were taken. Above 1600, I had only two observations and the results of these for $H = 2090$ and $H = 2590$ appear at the end of the table. The observations for $H = 707$ and $H = 1208$ were the only ones in these neighborhoods and are given as they stand. The value of I for $H = 2590$ happened to come out 1693, but a study of the observations shows that this number is really

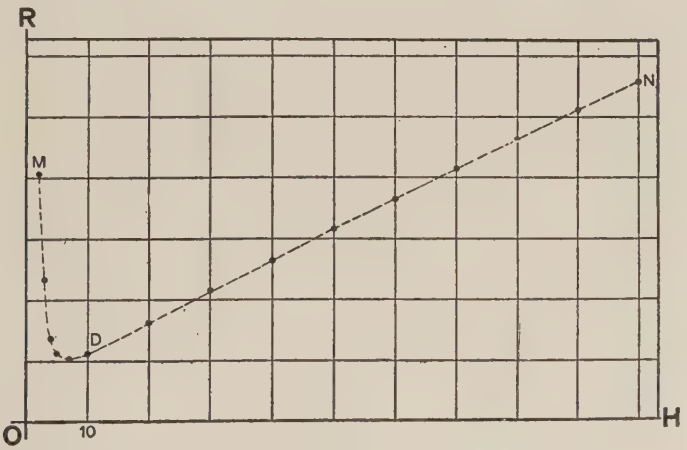


FIGURE 8. — This curve shows the reluctivity of the compressed steel between $H = 3$ and $H = 100$.

uncertain by about two units in the last place, for B is not determined in the fourth significant place.

An isthmus cut from this rod gave for a long series of observations

for values of H up to 16300, the same final value for I, within a fraction of 1 per cent, as the results tabulated above.

In view of the importance of Fröhlich's Law for magnetic metals at commercial excitations, figure 5, which shows the reluctivity of this

TABLE III

HALF-INCH ROD OF NORWAY IRON IN LONG SOLENOID

H	B	H	B
0	0		
1.0	1960	24.	16180
2.0	6950	28.	16320
3.0	9780	30.	16400
4.0	11380	40.	16650
5.0	12400	50.	16920
6.0	13080	60.	17180
7.0	13640	70.	17400
8.0	14140	80.	17600
10.	14800	100.	17940
12.	15200	120.	18220
14.	15460	160.	18800
16.	15620	200.	19100
18.	15780	240.	19840
20.	15960	280.	20200

TABLE IV

ANNEALED HALF-INCH NORWAY IRON ROD IN LONG SOLENOID

H	B	H	B
0	0		
0.2	190	9.5	14800
0.5	395	10.	14940
0.8	1120	11.	15100
1.0	2160	12.	15360
1.5	4600	14.	15540
2.0	6600	16.	15700
2.5	8240	18.	15900
3.0	9480	20.	16040
3.5	10460	25.	16320
4.0	11280	30.	16520
4.5	11980	35.	16740
5.0	12560	40.	16920
5.5	13000	45.	17100
6.0	13400	50.	17220
6.5	13700	60.	17450
7.0	13900	70.	17630
7.5	14100	80.	17820
8.0	14300	90.	18020
8.5	14490	100.	18210
9.0	14660	105.	18300

specimen of soft Bessemer steel for values of H up to 160, is of some interest.

At low excitations, the determinations of B for this rod were a little

TABLE V

ANNEALED NORWAY IRON ROD IN MASSIVE YOKE

Free length about 80 mm.

H	B	I
295	21120	1657
350	21550	1687
400	21850	1707
450	22100	1723
500	22220	1729
600	22480	1741
700	22630	1745
800	22770	1748
900	22880	1749
1000	23000	1750
1500	23500	1751
1800	23810	1751
2000	24010	1751
2350	24360	1751

TABLE VI

HALF-INCH ROD OF COMPRESSED SHAFTING IN LONG SOLENOID

H	B	H	B
0	0	13.	10400
0.5	70	14.	10690
1.0	155	15.	10970
1.5	290	20.	12300
2.0	490	25.	13260
2.5	800	30.	13950
3.0	1280	40.	15150
3.5	2010	50.	15850
4.0	2880	60.	16480
4.5	3210	70.	16950
5.0	4450	80.	17320
5.5	5120	90.	17640
6.0	5730	100	17900
6.5	6270	120	18200
7.0	6780	140	18560
7.5	7200	160	18800
8.0	7610	180	19080
8.5	8020	200	19320
9.0	8380	240	19700
9.5	8690	280	20000
10.	9010	320	20360
11.	9540	360	20600
12.	10100	400	20820

uncertain because of the difficulty of demagnetizing the specimen completely. Investigation showed that there were very slight differences of temper at different parts of the rod, and it seemed best to have the iron thoroughly annealed. This process increased the permeability for almost all excitations, very materially, as a comparison of Tables III and IV will show.

TABLE VII

SHORT PIECE OF HALF-INCH COMPRESSED SHAFTING MAGNETIZED IN MASSIVE YOKE

Free length about 80 mm.

H	B	I
300	20220	1585
370	20630	1612
400	20820	1625
500	21200	1647
600	21500	1663
700	21670	1669
800	21810	1672
1000	22080	1678
1200	22340	1682
1600	22800	1687
1854	23080	1690

TABLE VIII

H	B	μ	R	I
300	17470	58.2	0.017	1364
350	17750	50.7	0.020	1385
400	17980	44.9	0.022	1399
500	18370	36.7	0.027	1422
600	18710	31.2	0.032	1441
800	19240	24.0	0.042	1468
1000	19680	19.7	0.051	1487
1200	20050	16.7	0.060	1500
1400	20440	14.6	0.068	1515
1600	20790	13.0	0.077	1527
2000	21400	10.1	0.094	1544
2500	22010	8.8	0.114	1549

Figures 6 and 7 show the reluctivity of this iron at different excitations.

Two other half-inch specimens of very pure Norway iron turned from two-inch bars obtained from different sources gave as maximum values of the magnetization vector (I) 1732 and 1738. These were

in the condition in which they were bought and had not been specially annealed.

Figure 8 shows the reluctivity of this steel for excitations below $H = 100$.

Table VIII gives the results of an interesting study made by Mr. Coulson of the magnetic properties of a piece of Kidd's polished tool steel, for excitations between $H = 250$ and $H = 2500$.

The line obtained by plotting the reluctivity (R) against H has only a very gentle curvature between these limits for this annealed tool steel.

I wish to express my obligations to the Trustees of the Bache Fund of the National Academy of Sciences for the loan of some of the apparatus used in making the measurements recorded in this paper.

XIII

ON THE MAGNETIC PROPERTIES AT HIGH EXCITATIONS OF A REMARKABLY PURE SPECIMEN OF SOFT NORWAY IRON [1]

Some months ago an electromagnet was made for special use in the Jefferson Laboratory which had the form of a toroid uniformly wound with insulated wire for nineteen-twentieths of its perimeter. The core was of stout iron rod bent into the shape of a ring — complete except for a gap one centimeter wide. The mean diameter of the core was about fifty centimeters and a meridian section of the iron had an area of about twenty square centimeters. The exciting coil was made of about thirty kilograms of No. 10 B. & S. wire and the magnet had the general appearance indicated by figure 1, although the turns of wire which show in the photograph belong to a short test coil outside the winding proper.

It is evident that, under the most favorable circumstances, the leakage in the case of a magnet of these dimensions must be very large, but when this magnet was tried its performance fell so far below what, according to any known experience, it ought to have been, that it was thought best to have the iron tested both chemically and magnetically in the hope that the information thus procured might prove valuable in future designing. This seemed the more desirable since the core had been obtained by Professor Trowbridge, the Director of the Laboratory, in response to his inquiry for the very best brand of soft Norway iron to be had in the market.

The chemical analysis made by Mr. Emile Raymond Riegel showed this commercial iron to be of an extraordinary purity. The tests for nickel, cobalt, manganese, tungsten, and for "Groups IV and V" were all negative. There was less than 0.03 per cent of carbon, less than 0.047 per cent of phosphorus, less than 0.03 per cent of silicon, and less than 0.003 per cent of sulphur. The iron dissolved violently

[1] *American Journal of Science*, 4th series, vol. xxviii, no. 163, July, 1909.

in slightly diluted HNO_3, and when the residue had been dissolved for carbon, a mere discoloration of the beaker remained.

There was nothing, therefore, in the composition of the iron core to account for the comparative uselessness of the magnet.

The response of this remarkable iron to magnetic excitation was equally satisfactory, and the present report describes briefly deter-

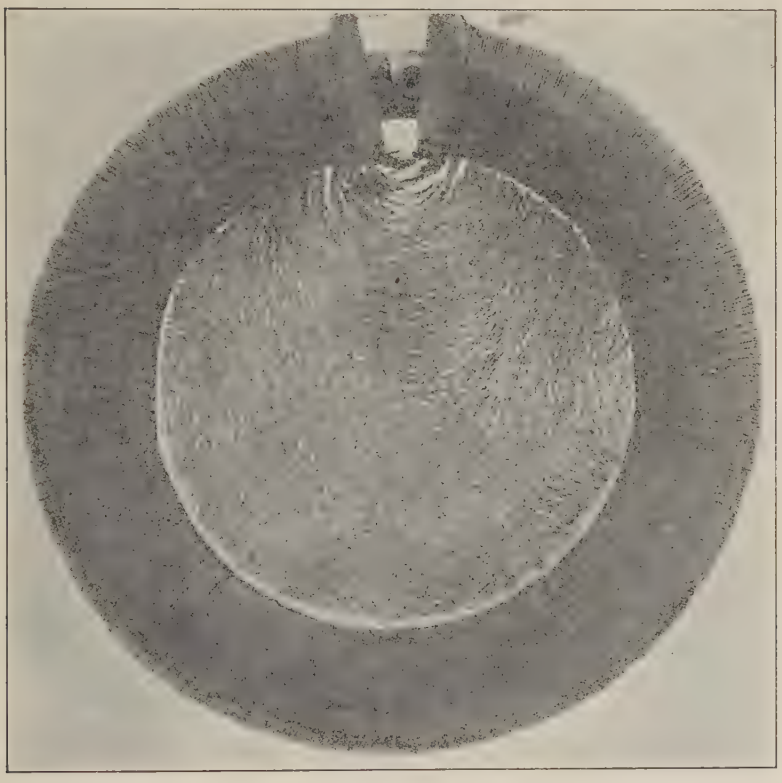


FIGURE 1

minations of the permeabilities of two pieces of it under very strong magnetizing fields. The work was done by Mr. John Coulson and myself, and was extremely troublesome because only a short stout piece of the iron used in making the core was available. From this a rod 1.26 centimeters in diameter and about 30 centimeters long was turned by Mr. G. W. Thompson, the mechanician of the Jefferson

Laboratory, and this rod was tested in various ways in the yoke represented in Figure 2.[1] Jaws of various shapes were tried and different ways of making the joints between the jaws and the test-piece. Usually under strong magnetic excitation, between the jaws of the yoke, there was a sensible leakage of lines of induction through the surface of the specimen into the air, and the field in the air about the rod was far from uniform in any available portion. We found eventually, however, that if a piece of the rod of about 80 millimeters free length, with tapered ends, was inserted into holes in the ends of the conical jaws represented in Figure 3,[1] the lines of force in the air just about the specimen near its center were for a considerable distance practically parallel to the axis of the rod, and that the value of H in the air in this region was sensibly equal to the value of the same quantity in the rod.

After a specimen of this standard length had been accurately fitted to the jaws by Mr. Thompson, the central portion of the iron rod was given a very thin coat of shellac varnish and two test coils, each consisting of twenty turns of very fine well insulated wire, were wound side by side in a single layer over the rod and these extended over rather more than a centimeter of the length of the specimen near its center. These coils were first tested against each other to find out whether they were practically alike, and then — if this condition was satisfied — both together in series formed the inner test coil (K). The outer test coil (L) was wound in a single layer on a very thin shell of boxwood which had been seasoning for many years. After corrections had been made for the thickness of the wire of the test coils and of its insulation, it was possible to compute from the measured change of induction flux through K and L due to a reversal of the current in the exciting circuit of the yoke, corresponding values of H and B.

The ballistic galvanometer used in this work had a period so long that no appreciable error was caused by the fact that several seconds were necessary to bring about a complete reversal of magnetization in the magnetic circuit. The galvanometer has been described under the letter V, in the Proceedings of the American Academy of Sciences in December of last year.

The test coils were wound by Mr. Coulson, who has helped in all the work.

[1] Figs. 2 and 3 are identical with Figs. 1 and 2 on pp. 211 and 212 respectively.

The iron of which the magnet core described above was made is here denoted by the letter P, while Q denotes a similar very pure specimen of Norway iron obtained from a new source.

TABLE I

SPECIMEN OF NORWAY IRON (P) MAGNETIZED IN MASSIVE YOKE

Free length about 80 millimeters, diameter 12.67 millimeters

H	*B*	*I*
150	19160	1513
200	19920	1566
300	21040	1650
400	21660	1692
500	21920	1705
600	22130	1713
700	22300	1720
800	22450	1723
1000	22720	1729
1200	22940	1730
1400	23180	1731
1600	23380	1732
2000	23780	1733
2500	24280	1733

The maximum value of I seems to be in the vicinity of 1733, and for large values of the excitation corresponding values of H and B may be computed by means of the equation $B = H + 21780$.

This record shows conclusively that the magnetic permeability of this iron under strong excitation is extraordinarily high and that the failure of the magnet mentioned at the beginning of this report was not due to poor material in the core. The real source of the difficulty is disclosed by an examination of the diagram shown in Figure 4. This was obtained by sprinkling iron filings upon a horizontal piece of cardboard which rested on the toroid as it lay upon the floor and carried a heavy current. Although the cardboard was not favorably placed, there are evidences that at least ten consequent poles were created between the ends of the core when it was strongly excited. When the exciting current was reversed these poles changed sign, but in many places outside the exciting coil the direction of the field was always opposed to what it would be if these consequent poles did not exist. This core has been annealed as well as the maker could do it, after it had been bent into shape, but the process demands great skill

and, as is well known, soft Norway iron is very likely to acquire slight differences of temper due to unequal heating in the forge fire.

Table II exhibits the results of some observations made upon a half-inch rod of Norway iron (R), when magnetized in a uniformly wound solenoid. The rod was about ten feet long. When it was purchased this iron was very soft as is shown by the numbers in the second

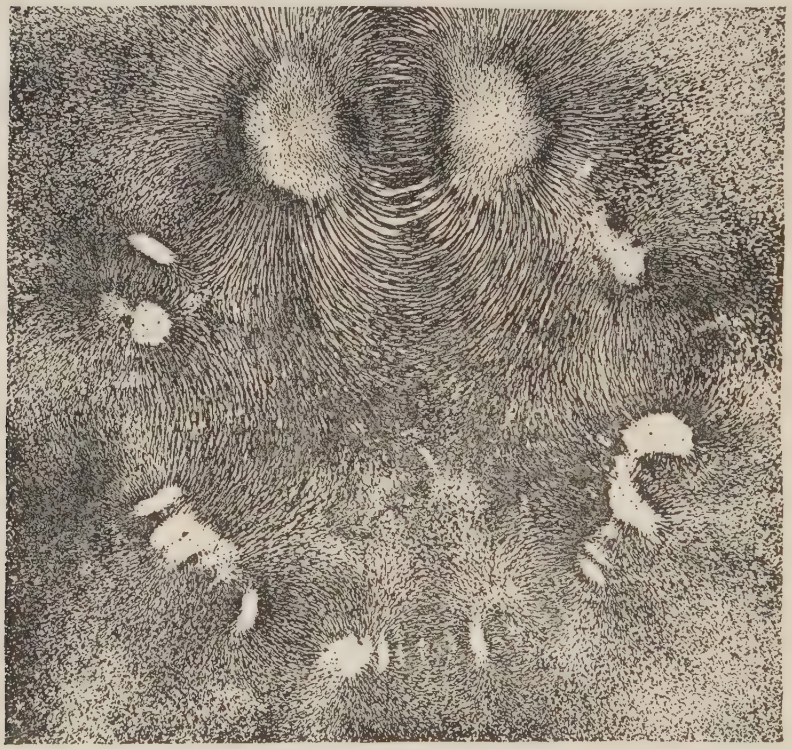


FIGURE 4

column, which gives the values of the induction (B) corresponding to the values of H in the first column. When, however, the rod had been again subjected by Mr. Thompson to an elaborate annealing process, its permeability had been somewhat increased as appears from the values of B exhibited in the third column.

Specimen Q, like specimen P, was cut from a bar of the best Norway iron two inches in diameter, but the two bars came from different

dealers. These irons seem to be nearly alike in temper and in composition.

From $H = 1100$ up to $H = 2450$, the observed values of I differ on the average from their mean by about one-sixth of one per cent only.

TABLE II

NORWAY IRON ROD (R) MAGNETIZED IN THE LONG SOLENOID

Length about 300 centimeters, diameter 12.67 millimeters

H	B (Before the iron had been annealed)	B (After the rod had been annealed)
5	12400	12560
10	14800	14940
14	15460	15540
20	15960	16040
30	16400	16520
40	16650	16920
50	16920	17220
60	17180	17450
70	17400	17630
80	17600	17820
100	17940	18210

TABLE III

SPECIMEN OF NORWAY IRON (Q) MAGNETIZED IN MASSIVE YOKE

Free length about 80 millimeters, diameter 12.67 millimeters

H	B	I
300	20530	1610
400	21110	1648
600	22020	1704
700	22300	1719
800	22510	1728
1000	22800	1735
1200	23020	1737
1400	23240	1738
1650	24240	1738
2000	23840	1738
2400	23490	1738

For high excitations, corresponding values of H and B may be obtained from the equation $B = H + 21840$.

Table IV shows the results of some determinations of the maximum value of I made upon an isthmus piece of the iron P after it had been subjected to an annealing process lasting about 48 hours and was therefore extremely soft.

TABLE IV

SPECIALLY ANNEALED ISTHMUS PIECE OF NORWAY IRON (P)

Cross sections of isthmus 0.2050 square centimeters; mean area of inner test coil 0.2230 square centimeters; mean area of outer test coil 0.5025 square centimeters

Exciting Current	*H*	*B*	*I*
1.00	5920	28580	1799
4.07	12370	34900	1794
5.58	13720	36210	1790
9.90	16000	38530	1793
23.00	18130	40780	1802
31.00	18810	41400	1797

For a current of about 55 amperes a value $B = 42200$ was reached but the current fell so rapidly that H could not be accurately determined. In this case the excitation was upwards of 160,000 ampere-turns.

It is interesting to compare this remarkable value for the maximum intensity of magnetization with that obtained for a specimen of the iron R, after it had been thoroughly annealed.

TABLE V

ANNEALED NORWAY IRON (R) IN MASSIVE YOKE

Free length about 80 millimeters

H	*B*	*I*
800	22770	1748
900	22880	1749
1000	23000	1750
1500	23500	1751
1800	23810	1751
2000	24010	1751
2350	24360	1751

XIV

THE CONCEPTION OF THE DERIVATIVE OF A SCALAR POINT FUNCTION WITH RESPECT TO ANOTHER SIMILAR FUNCTION [1]

In modern treatises on Mathematical Physics it is customary to define the derivative of a scalar function, taken at a given point in space in a given direction, in a manner which emphasizes the fact that this derivative is an invariant of a transformation of coördinates. According to this definition,[2] if through the point P a straight line be drawn in a fixed direction (s), if on this line a point P' be taken near P so that PP' has the direction s, and if u_P, $u_{P'}$ be used to represent the values at these points of the scalar point function u, then if the ratio

$$\frac{u_{P'} - u_P}{PP'} \tag{1}$$

approaches a limit as P' approaches P, this limit is called the derivative of u, at P, in the direction s. If u happens to be defined in terms of a system of orthogonal Cartesian coördinates, x, y, z, and has continuous derivatives with respect to these coördinates everywhere within a certain region, the limit just mentioned exists in this region and its value is

$$\frac{\partial u}{\partial x} \cdot \cos(x, s) + \frac{\partial u}{\partial y} \cdot \cos(y, s) + \frac{\partial u}{\partial z} \cdot \cos(z, s). \tag{2}$$

Of all the numerical values which the derivative of u can have at a given point, the greatest is to be found by making s normal to the

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlv, no. 12, April, 1910.

[2] Hamilton, *Elements of the Theory of Quaternions;* Tait, *Elementary Treatise on Quaternions;* Gibbs, *Vector Analysis;* Maxwell, *Treatise on Electricity and Magnetism;* Webster, *Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies;* Jeans, *Mathematical Theory of Electricity and Magnetism;* Lamé, *Leçons sur les Coordonnées Curvilignes;* Peirce, *Theory of the Newtonian Potential Function; Generalized Space Differentiation of the Second Order;* Czuber, *Wienerberichte*, vol. ci_{λ}, 1417 (1892); Boussinesq, *Cours d'Analyse Infinitésimale;* H. Weber, *Die Partiellen Differential-Gleichungen der Mathematischen Physik.*

level surface of u which passes through the point. This maximum value,

$$\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]^{\frac{1}{2}}, \tag{3}$$

is usually regarded as the value at the point of a vector point function called the gradient vector of u, the lines of which cut orthogonally the level surfaces of u, and the components of which parallel to the coordinate axes are

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}. \tag{4}$$

This vector is, of course, lamellar.

The value of the tensor of the gradient vector is often called simply the "gradient" of u and is denoted by h_u. If at any point a straight line be drawn in the direction (n) normal to the level surface of u, in the sense in which u increases, and if a length h_u be laid off on this line, the projection,

$$h_u \cdot \cos(n, s), \tag{5}$$

of this length on any other direction (s) is numerically equal to the derivative of u in the direction s.

Most physical quantities — such as temperature, barometric pressure, density, inductivity — present themselves to the investigator as single valued point functions, which, except perhaps at one or more given surfaces of discontinuity, are differentiable in the sense just considered.

It is often desirable to differentiate a scalar function, u, at a point, in the direction in which another scalar function, v, increases fastest, and if (u, v) represents the angle between the gradient vectors of u and v at the point, the derivative is evidently equal to

$$h_u \cdot \cos(u, v). \tag{6}$$

It frequently happens that in a question of maxima and minima, one wishes to determine the greatest (or the smallest) value which a quantity U may have, subject to the condition that another quantity V shall have a given value (V_0). If these quantities can be represented by point functions, the problem geometrically considered requires one to find the parameter of a surface of the constant U family,

which is tangent to the surface of the V family upon which V is everywhere equal to V_0; but at the point of tangency, the derivative of the function U in any direction in the tangent plane of the V surface is zero, that is, the normals to the U and V surfaces coincide, so that

$$\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial v}{\partial x}} = \frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial v}{\partial y}} = \frac{\dfrac{\partial u}{\partial z}}{\dfrac{\partial v}{\partial z}}, \tag{7}$$

and these familiar equations usually furnish some general information about the problem independent of the value of V_0. As an extremely simple example we may take the familiar problem concerning the relative dimensions of an open tank of square base ($x \times x$) and height y,

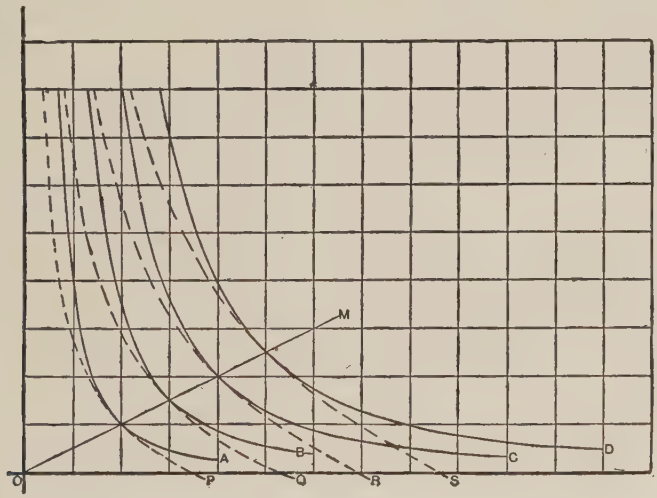


FIGURE 1

which shall hold a given quantity ($V = x^2 \cdot y$) of water and have the smallest wet surface ($U = x^2 + 4xy$). Here we have the curve D of the V family, which has the given parameter, V_0, and are required to find that member of the P, Q, R, S family which touches D. The equation (7) becomes in this case $2y = x$, and it appears (Figure 1) that the curves of the two families which pass through any point of the line OM are at that point tangent to each other.

It is sometimes necessary to differentiate a point function, u, at a point P, in the direction of the line through the point, along which two other point functions, v, w, are constant; that is, along the line $v = v_P$, $w = w_P$. If

$$L = \begin{vmatrix} \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}, \quad M = \begin{vmatrix} \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial z} & \frac{\partial w}{\partial x} \end{vmatrix}, \quad N = \begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} \tag{8}$$

and if $R^2 = L^2 + M^2 + N^2$ — which is equal to $h_v^2 \cdot h_w^2$, if v and w are orthogonal — this direction is defined by the cosines L/R, M/R, N/R, and the derivative required is

$$\frac{1}{R}\left(L \cdot \frac{\partial u}{\partial x} + M \cdot \frac{\partial u}{\partial y} + N \cdot \frac{\partial u}{\partial z}\right). \tag{9}$$

If the maxima and minima of the function $u = f(x, y, z)$ are to be found under the condition that the functions v, w shall have given numerical values, the derivative of u taken in the direction in which v and w are constant must be made to vanish. Thus, if

$$u = x^2 + y^2 + z^2,$$

and if the conditions are

$$xyz = c^3 \text{ and } x + y = d,$$

equation (9) yields immediately the required relation

$$(xy + z^2)(y - x) = 0.$$

When $f'(u)$ is positive, the direction of the gradient vector of $f(u)$ coincides with that of the gradient vector of u itself: these directions are opposed when $f'(u)$ is negative. The tensors of both vectors are always positive. If

$$w = f(u), \; h_w^2 = [f'(u)]^2 \cdot h_u^2, \text{ and } \cos(w, s) = \cos(u, s):$$

in particular, when

$$w = 1/u, \; h_w = h_u/u^2 \text{ and } \cos(w, s) = -\cos(u, s),$$

so that

$$\frac{\partial}{\partial s}\left(\frac{1}{u}\right) = -\frac{\partial u}{\partial s} \cdot \frac{1}{u^2}.$$

If u is the distance (r) to a point on a curve (s) from a fixed point outside the curve,

$$\frac{\partial r}{\partial s} = +\cos(s, r), \frac{\partial}{\partial s}\left(\frac{1}{r}\right) = -\frac{\cos(s, r)}{r^2}.$$

Any function of the complex variable $(ax + by + iz\sqrt{a^2+b^2})$ has a gradient identically equal to zero, but every differentiable real point function has a gradient in general different from zero. The gradient of a function may be constant throughout a region of space: if the gradient of u is constant, the surfaces upon each of which u is constant form a parallel system. If the gradient of a function, u, is either constant or expressible in terms of u, any differentiable function of u has a gradient either constant or expressible in terms of u. If the gradient of u is expressible in terms of u alone $[h_u = f(u)]$, it is possible to form a function, $a\int\frac{du}{f(u)}$, of u the gradient of which shall be constant. If h_u is neither constant nor expressible in terms of u, no function of u exists the gradient of which is expressible in terms of u. The functions $u = \sin(x + y + z)$, $v = \sin(x + 2y - 3z)$, $w = \sin(5x - 4y - z)$ illustrate the fact that the gradient of each of three orthogonal point functions may be expressible in terms of the function itself.

If the gradient of each of two orthogonal point functions, u, v, were expressible as the product of a function of u and a function of v, so that $h_u = U_1 \cdot V_1$, and $h_v = U_2 \cdot V_2$, it would be possible to form two functions $\left[\int\frac{du}{U_1}, \int\frac{dv}{V_2}\right]$ of u alone and of v alone, respectively, the gradient of each of which would be expressible in terms of the other. If the gradient vectors of two functions have the same direction at every point of space, one of these functions is expressible in terms of the other. If the gradients of two real functions, u, v, are everywhere equal while the directions of their gradient vectors are different,

$$\frac{\partial(u+v)}{\partial x}\cdot\frac{\partial(u-v)}{\partial x} + \frac{\partial(u+v)}{\partial y}\cdot\frac{\partial(u-v)}{\partial y} + \frac{\partial(u+v)}{\partial z}\cdot\frac{\partial(u-v)}{\partial z} = 0, \quad (10)$$

and the functions $[u + v]$, $[u - v]$ are orthogonal, as are $F(u + v)$, $f(u - v)$, where F and f are any differentiable functions. If u and v are orthogonal functions, the functions $[F(u) + f(v)]$, $[F(u) - f(v)]$ have gradients numerically equal to each other at every point.

Two scalar point functions, the level surfaces of which are neither coincident nor orthogonal, may have gradients each of which is expressible in terms of the other: the gradient of $v = \frac{4}{3}x^3 - 4xy^2$ is equal at every point of the xy plane to the square of the gradient of $u = x^2 - y^2$. If u and v are orthogonal functions of x and y, the product of their gradients is equal to the Jacobian,

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}.$$

The differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = k^2,$$

which leads to systems of parallel surfaces, is of standard form. Its complete integral is

$$u = ax + by + z\sqrt{k^2 - a^2 - b^2} + d,$$

where a, b, d are arbitrary constants, and from this the general integral may be obtained in the usual manner.

If a direction s be determined at every point of a given region, T, by some law, the derivative of the function u becomes itself a scalar point function in T, and if this is differentiable, it may be differentiated at any point in any direction, say s. It is usually convenient to define s by means of three scalar point functions, l, m, n, the sum of the squares of which is identically equal to unity, and which represent the direction cosines of s. In this connection it is well to notice that if s has the direction at P of the tangent of a continuous curve which passes through the point, if P' be a point near P on the tangent and P'' a point near P on the curve, and if U is any differentiable scalar point function,

$$\frac{U_{P''} - U_P}{PP''}, \qquad \frac{U_{P'} - U_P}{PP'}$$

have the same limit, as P' and P'' approach P, as that which has been defined as the derivative of U at P in the direction s. If, then, $\partial u/\partial s$ is differentiable

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right) = \frac{\partial}{\partial x}\left(l\cdot\frac{\partial u}{\partial x} + m\cdot\frac{\partial u}{\partial y} + n\cdot\frac{\partial u}{\partial z}\right)$$
$$= l\cdot\frac{\partial^2 u}{\partial x^2} + m\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + n\cdot\frac{\partial^2 u}{\partial x\cdot\partial z} + \frac{\partial u}{\partial x}\cdot\frac{\partial l}{\partial x}$$
$$+ \frac{\partial u}{\partial y}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial n}{\partial x}, \qquad (11)$$

and

$$\frac{\partial^2 u}{\partial s^2} = l^2\cdot\frac{\partial^2 u}{\partial x^2} + m^2\cdot\frac{\partial^2 u}{\partial y^2} + n^2\cdot\frac{\partial^2 u}{\partial z^2} + 2lm\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + 2mn\cdot\frac{\partial^2 u}{\partial y\cdot\partial z}$$
$$+ 2ln\cdot\frac{\partial^2 u}{\partial x\cdot\partial z} + \frac{\partial u}{\partial x}\left(l\cdot\frac{\partial l}{\partial x} + m\cdot\frac{\partial l}{\partial y} + n\cdot\frac{\partial l}{\partial z}\right)$$
$$+ \frac{\partial u}{\partial y}\left(l\cdot\frac{\partial m}{\partial x} + m\cdot\frac{\partial m}{\partial y} + n\cdot\frac{\partial m}{\partial z}\right)$$
$$+ \frac{\partial u}{\partial z}\left(l\cdot\frac{\partial n}{\partial x} + m\cdot\frac{\partial n}{\partial y} + n\cdot\frac{\partial n}{\partial z}\right). \qquad (12)$$

If s' is a direction defined by the cosines l', m', n',

$$\frac{\partial^2 u}{\partial s'\cdot\partial s} = ll'\cdot\frac{\partial^2 u}{\partial x^2} + mm'\cdot\frac{\partial^2 u}{\partial y^2} + nn'\cdot\frac{\partial^2 u}{\partial z^2}$$
$$+ (lm' + l'm)\frac{\partial^2 u}{\partial x\cdot\partial y} + (mn' + m'n)\frac{\partial^2 u}{\partial y\cdot\partial z} + (nl' + n'l)\frac{\partial^2 u}{\partial z\cdot\partial x}$$
$$+ \frac{\partial u}{\partial x}\left(l'\cdot\frac{\partial l}{\partial x} + m'\cdot\frac{\partial l}{\partial y} + n'\cdot\frac{\partial l}{\partial z}\right) + \frac{\partial u}{\partial y}\left(l'\cdot\frac{\partial m}{\partial x} + m'\cdot\frac{\partial m}{\partial y} + n'\cdot\frac{\partial m}{\partial z}\right)$$
$$+ \frac{\partial u}{\partial z}\left(l'\cdot\frac{\partial n}{\partial x} + m'\cdot\frac{\partial n}{\partial y} + n'\cdot\frac{\partial n}{\partial z}\right), \qquad (13)$$

and it is clear that the order of differentiation is usually not commutative. Derivatives of this kind are often found in differential equations of orders higher than the first which define functions in terms of simple curvilinear coördinates.

If for instance spherical coördinates are to be used, the second derivative of u taken in the direction in which θ increases fastest is

$$\frac{\partial^2 u}{\partial x^2}\cdot\cos^2\theta\cos^2\phi + \frac{\partial^2 u}{\partial y^2}\cdot\cos^2\theta\sin^2 y + \frac{\partial^2 u}{\partial z^2}\cdot\sin^2\theta + \frac{2\partial^2 u}{\partial x\cdot\partial y}\cdot\cos^2\theta\sin\phi\cos\phi$$
$$- \frac{2\partial^2 u}{\partial x\cdot\partial z}\cdot\sin\theta\cos\theta\cos\phi - \frac{2\partial^2 u}{\partial y\cdot\partial z}\cdot\sin\theta\cos\theta\sin\phi$$
$$- \frac{\partial u}{r\cdot\partial x}\cdot\sin\theta\cos\phi - \frac{\partial u}{r\cdot\partial y}\cdot\sin\theta\sin\phi - \frac{\partial u}{r\cdot\partial z}\cdot\cos\theta \qquad (14)$$

and this, which contains derivatives of the first order, is in sharp contrast to the second derivative of u taken in the direction r, which is,

$$\frac{\partial^2 u}{\partial x^2}\cdot\sin^2\theta\cos^2\phi+\frac{\partial^2 u}{\partial y^2}\cdot\sin^2\theta\sin^2\phi+\frac{\partial^2 u}{\partial z^2}\cdot\cos^2\theta+\frac{2\partial^2 u}{\partial x\cdot\partial y}\cdot\sin^2\theta\sin\phi\cos\phi$$
$$+\frac{2\partial^2 u}{\partial y\cdot\partial z}\cdot\sin\theta\cos\theta\sin\phi+\frac{2\partial^2 u}{\partial z\cdot\partial x}\cdot\sin\theta\cos\theta\cos\phi. \qquad (15)$$

Sometimes s and s' are fixed directions so that l, m, n, l', m', n', are constants throughout T, and in this case the coefficients of $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ in (12) and (13) vanish. The mutual potential energy W, of two magnetic elements, M, M', of moments, m, m', can be written in the form

$$m\cdot m'\frac{\partial^2}{\partial s\cdot\partial s'}\left(\frac{1}{r}\right), \qquad (16)$$

where r is the distance MM' and s, s' are the directions of the axes of the elements. The force (due to the second magnet) which tends to move the first magnet in the direction of its own axis is then

$$-m\cdot m'\frac{\partial^3}{\partial s\cdot\partial s\cdot\partial s'}\left(\frac{1}{r}\right) \qquad (17)$$

and these differentiations assume that the direction cosines of s and s' are constants.

In general, if s is the direction perpendicular to the level surface of u, and if h is the scalar point function which gives the value of $\partial u/\partial s$,

$$\frac{\partial^2 u}{\partial s^2}=\left(\frac{\partial h}{\partial x}\cdot\frac{\partial u}{\partial x}+\frac{\partial h}{\partial y}\cdot\frac{\partial u}{\partial y}+\frac{\partial h}{\partial z}\cdot\frac{\partial u}{\partial z}\right)\Big/h. \qquad (18)$$

In the case of oblique Cartesian coördinates in a plane, x increases fastest in a direction which is not perpendicular to the line along which it is constant. If the angle between the coördinate axes is ω,

$$\frac{\partial u}{\partial x}=h_u\cdot\cos(x,h_u),\quad \frac{\partial u}{\partial y}=h_u\cdot\cos(y,h_u),\quad \frac{\partial u}{\partial s}=h_u\cdot\cos(s,h_u),$$
$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\sin(y,s)}{\sin\omega}+\frac{\partial u}{\partial y}\cdot\frac{\sin(x,s)}{\sin\omega} \qquad (19)$$

It is frequently necessary to differentiate one point function, U, with respect to another, u, and the process usually appears in the form of

a kind of partial differentiation. If, for instance, U is to satisfy a differential equation in terms of a set of orthogonal curvilinear coördinates of which u is one, the derivatives of U with respect to u are to be taken on the assumption that the other coördinates remain constant. This large subject has been treated exhaustively in the many works on orthogonal coördinates which have been published since Lamé's classical treatise [1] appeared.

Given a function, u, it is, however, not generally possible to find a system of orthogonal functions of which u shall be one, and it is often convenient for a physicist to differentiate a physical function, U, with respect to another, u, without considering the existence of any other related functions. A physical point function has a value at every point in space which is not altered by changing the system of coördinates which fix the position of the point, and it is well to define the derivative of U with regard to u in a manner which shall emphasize the fact that the derivative is an invariant of a change of coördinates and which shall not assume that two functions (v, w) can be found orthogonal to each other and to u. When U and u are considered by themselves and not regarded as coördinated of necessity with other similar quantities, it is usually, if not always, the case that a "normal" derivative [2] is required.

The *normal* derivative, at any point, P, of the differentiable scalar point function U, with respect to the differentiable scalar point function u, may be defined as the limit, when PP' approaches zero, of the ratio

$$\frac{U_{P'} - U_P}{u_{P'} - u_P}, \tag{20}$$

where P' is a point so chosen on the normal at P of the surface of constant u which passes through P, that $u_{P'} - u_P$ shall be positive. If (U, u) denotes the angle between the directions in which U and u increase most rapidly, the normal derivatives of U with respect to u and of u with respect to U may be written

[1] Lamé, *Leçons sur les Coordonnées Curvilignes et leur Diverses Applications;* Salvert, *Mémoire sur l'Emploi des Coordonnées Curvilignes;* Darboux, *Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes;* Goursat, *Cours d'Analyse Mathématique.*

[2] Peirce, *Short Table of Integrals, Theory of the Newtonian Potential Function; Generalized Space Differentiation of the Second Order.*

$$\frac{h_U}{h_u} \cdot \cos(U, u) \text{ and } \frac{h_u}{h_U} \cdot \cos(U, u). \tag{21}$$

If $h_U = h_u$ these derivatives are equal. An example of this is the equality of $\partial n/\partial r$ and $\partial r/\partial n$ in a familiar application of Green's Theorem, where n and r represent the normal distance from a given surface and the distance from a given fixed point respectively. If U and u happen to be expressed in terms of a set (x, y, z) of orthogonal Cartesian coördinates, the normal derivative of U with respect to u can be written

$$D_u U = \frac{\frac{\partial U}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial U}{\partial y} \cdot \frac{\partial u}{\partial y} + \frac{\partial U}{\partial z} \cdot \frac{\partial u}{\partial z}}{h_u^2}, \tag{22}$$

and it is easy to see that this is equal to the ratio of the derivatives of U and u taken in the direction in which u increases most rapidly.

It is occasionally instructive to use the conception of normal differentiation in studying some of the general equations of Physics: thus in the uncharged dielectric about an electric distribution, the potential function, V, is connected with the inductivity of the medium, μ, by the familiar equation

$$\frac{\partial}{\partial x}\left(\mu \cdot \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \cdot \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \cdot \frac{\partial V}{\partial z}\right) = 0, \tag{23}$$

in which μ is to be regarded as a point function discontinuous in general at each of a given set of surfaces at every point of which an equation of the form

$$\mu_1 \cdot \frac{\partial V}{\partial n_1} + \mu_2 \cdot \frac{\partial V}{\partial n_2} = 0 \tag{24}$$

is satisfied. Now (23) may be put into the form

$$\frac{\partial \log \mu}{\partial V} + \frac{\nabla^2 V}{h_V^2} = 0, \tag{25}$$

and according to Lamé's condition, the second term is a function of V only, if the level surfaces of V are possible level surfaces of a harmonic function.

It is easy to make from (25), by inspection, such simple deductions as those which follow in this paragraph. If V is harmonic, either the dielectric is made up of homogeneous portions separated from one another by equipotential surfaces, or the level surfaces of μ and of V are

everywhere perpendicular to each other. If V, though not harmonic, satisfies Lamé's condition $[\nabla^2(V)/h_V{}^2 = F(V)]$ the level surfaces of the inductivity are equipotential; and if the level surfaces of V and μ are identical, V satisfies Lamé's condition. If when the plates of a condenser are kept at given potentials, the level surfaces of the inductivity of the dielectric are equipotential, the value of the potential function in the dielectric would be unchanged if μ were changed to $\Omega \cdot \mu$, where Ω is any scalar point function orthogonal to V. If the continuous dielectric of a condenser in which the level surfaces of the inductivity, μ, are equipotential be changed so as to make the new potential function between the plates a function $[V' = f(V)]$ of the old, the new inductivity must satisfy an equation of the form $\mu' = \Omega.\mu/f'(V)$. If the V and the μ surfaces are neither coincident nor orthogonal, V cannot be harmonic, and if V is given and one value of the inductivity found, no other value of the inductivity with the same level surfaces as this can be found except by altering the old value at every point in a constant ratio. If V does not satisfy Lamé's condition, a new value of the inductivity found by multiplying the old value by any point function orthogonal to V, will yield the same value of V, but the level surfaces of the inductivity will be altered. If the V and the μ surfaces are not coincident, no change of the inductivity which leaves its surfaces unchanged can make these surfaces equipotential.

If a mass of fluid, the characteristic equation of which is of the form $p = f(\rho, T)$, is at rest under the action of a conservative field of force the components of which are X, Y, Z,

$$\frac{\partial p}{\partial x} = \rho \cdot X, \quad \frac{\partial p}{\partial y} = \rho \cdot Y, \quad \frac{\partial p}{\partial z} = \rho \cdot Z. \tag{26}$$

It follows immediately from these equations that p and V must be colevel, and the normal derivative of p with respect to V shows that equilibrium is impossible unless the distribution of temperature is such that the equipotential surfaces are also isothermal.

If the scalar point function, W, is expressed in terms of the three orthogonal point functions, u, v, w, the square of the gradient of W is well known to be equal to

$$h_u{}^2 \cdot \left(\frac{\partial W}{\partial u}\right)^2 + h_v \cdot \left(\frac{\partial W}{\partial v}\right)^2 + h_w{}^2 \cdot \left(\frac{\partial W}{\partial w}\right)^2 \cdot$$

If the vector point function Q is expressed in terms of u, v, w, the divergence of Q is equal to

$$h_u \cdot h_v \cdot h_w \left[\frac{\partial}{\partial u}\left(\frac{Q_u}{h_v \cdot h_w}\right) + \frac{\partial}{\partial v}\left(\frac{Q_v}{h_u \cdot h_w}\right) + \frac{\partial}{\partial w}\left(\frac{Q_w}{h_u \cdot h_v}\right)\right].$$

If the normal derivatives of u and v with respect to w be denoted by $D_w u$ and $D_w v$, it follows from the definition that

$$\begin{aligned} D_w(u+v) &= D_w u + D_w v, & D_w u^n &= n \cdot u^{n-1} \cdot D_w u, \\ D_w(u \cdot v) &= v \cdot D_w u + u \cdot D_w v, & D_w\left(\frac{u}{v}\right) &= \frac{v \cdot D_w u - u \cdot D_w v}{v^2}, \\ D_w f(u) &= f'(u) \cdot D_w(u). \end{aligned}$$

The normal derivative of u with respect to v is a scalar function which, if differentiable, has a normal derivative with respect to v, and since by definition

$$D_v x = \frac{D_x v}{h_v^2}, \tag{27}$$

$$D_v h_v = \frac{1}{h_v^2}\left\{ \frac{\partial h_v}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial h_v}{\partial y} \cdot \frac{\partial v}{\partial y} + \frac{\partial h_v}{\partial z} \cdot \frac{\partial v}{\partial z} \right\}, \tag{28}$$

we may write

$$\begin{aligned} D_v^2 u = {} & \frac{1}{h_v^4}\left\{ \frac{\partial^2 u}{\partial x^2} \cdot \left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial^2 u}{\partial y^2} \cdot \left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial z^2} \cdot \left(\frac{\partial v}{\partial z}\right)^2 \right\} \\ & + \frac{2}{h_v^4}\left\{ \frac{\partial^2 u}{\partial x \cdot \partial y} \cdot \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial y \cdot \partial z} \cdot \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial z} + \frac{\partial^2 u}{\partial z \cdot \partial x} \cdot \frac{\partial v}{\partial z} \cdot \frac{\partial v}{\partial x} \right\} \\ & + \frac{1}{h_v^3}\left\{ \frac{\partial u}{\partial x}\left(\frac{\partial h_v}{\partial x} - 2\frac{\partial v}{\partial x} \cdot \frac{\partial h_v}{\partial v}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial h_v}{\partial y} - 2\frac{\partial v}{\partial y} \cdot \frac{\partial h_v}{\partial v}\right) \right. \\ & \left. + \frac{\partial u}{\partial z}\left(\frac{\partial h_v}{\partial z} - 2\frac{\partial v}{\partial z} \cdot \frac{\partial h_v}{\partial v}\right) \right\} \end{aligned} \tag{29}$$

$$\begin{aligned} D_w D_v u = {} & \frac{1}{h_v^2 h_w^2}\left\{ \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial y^2} \cdot \frac{\partial w}{\partial y} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial z^2} \cdot \frac{\partial w}{\partial z} \cdot \frac{\partial v}{\partial z} \right\} \\ & + \frac{1}{h_v^2 \cdot h_w^2}\left\{ \frac{\partial^2 u}{\partial x \cdot \partial y}\left(\frac{\partial w}{\partial y} \cdot \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial v}{\partial y}\right) \right. \\ & + \frac{\partial^2 u}{\partial y \cdot \partial z}\left(\frac{\partial w}{\partial z} \cdot \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \cdot \frac{\partial v}{\partial z}\right) + \frac{\partial^2 u}{\partial x \cdot \partial z}\left(\frac{\partial w}{\partial z} \cdot \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial v}{\partial z}\right) \\ & + \frac{1}{h_v^2 \cdot h_w^2}\left\{ \frac{\partial u}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial y \cdot \partial x} \cdot \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x \cdot \partial z} \cdot \frac{\partial w}{\partial z}\right) \right. \\ & \left. - \frac{2}{h_v} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}\left(\frac{\partial h_v}{\partial x} \cdot \frac{\partial w}{\partial x} + \frac{\partial h_v}{\partial y} \cdot \frac{\partial w}{\partial y} + \frac{\partial h_v}{\partial z} \cdot \frac{\partial w}{\partial z}\right)\right\} \end{aligned}$$

$$\frac{1}{h_v^2 \cdot h_w^2}\left\{\frac{\partial u}{\partial y}\left(\frac{\partial^2 v}{\partial y^2}\cdot\frac{\partial w}{\partial y}+\frac{\partial^2 v}{\partial z\cdot\partial y}\cdot\frac{\partial w}{\partial z}+\frac{\partial v}{\partial y\cdot\partial x}\cdot\frac{\partial w}{\partial y}\right)\right.$$
$$\left.-\frac{2}{h_v}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\left(\frac{\partial h_v}{\partial y}\cdot\frac{\partial w}{\partial y}+\frac{\partial h_v}{\partial z}\cdot\frac{\partial w}{\partial z}+\frac{\partial h_v}{\partial x}\cdot\frac{\partial w}{\partial x}\right)\right\}$$
$$\frac{1}{h_v^2 \cdot h_w^2}\left\{\frac{\partial u}{\partial z}\left(\frac{\partial^2 v}{\partial z^2}\cdot\frac{\partial w}{\partial z}+\frac{\partial^2 v}{\partial x\cdot\partial z}\cdot\frac{\partial w}{\partial x}+\frac{\partial v}{\partial z\cdot\partial y}\cdot\frac{\partial w}{\partial y}\right)\right.$$
$$\left.-\frac{2}{h_v}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\left(\frac{\partial h_v}{\partial z}\cdot\frac{\partial w}{\partial z}+\frac{\partial h_v}{\partial x}\cdot\frac{\partial w}{\partial x}+\frac{\partial h_v}{\partial y}\cdot\frac{\partial w}{\partial y}\right)\right\} \qquad (30)$$

It is evident that $D_v D_w u$ is usually quite different from $D_w D_v u$.

In the transformation of a partial differential equation from one set of independent variables to another set which does not form an orthogonal system, derivatives occur which are not normal in the sense of the last paragraphs. If a mass of fluid is in motion under the action of given forces, it is usually convenient either to express the orthogonal coördinates of a particle which at the time t has the position (x, y, z) in terms of t and the coördinates x_0, y_0, z_0, which the same particle had at the origin of time, or to express x_0, y_0, z_0, as functions of x, y, z, t.

$$x_0 = f_1(x, y, z, t), \quad y_0 = f_2(x, y, z, t), \quad z_0 = f_3(x, y, z, t). \qquad (31)$$

In this case, it frequently happens that the level surfaces of f_1, f_2, f_3, are not orthogonal. According as we use the "historical" or the "statistical" method of studying the motion, we shall express the pressure and the density in terms of x_0, y_0, z_0, t, or in terms of x, y, z, t. Suppose the second method to have been chosen, and $\partial p/\partial x$ to have been found by the aid of Euler's Equations of Motion and the Equation of Continuity, and suppose that $\partial p/\partial x_0$ is needed. We shall then have

$$\frac{\partial p}{\partial x_0} = \frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial x_0}+\frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial x_0}+\frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial x_0}. \qquad (32)$$

If with the help of (31) we find the values of the determinants

$$L = \begin{vmatrix} \frac{\partial y_0}{\partial y} & \frac{\partial y_0}{\partial z} \\ \frac{\partial z_0}{\partial y} & \frac{\partial z_0}{\partial z} \end{vmatrix}, \quad M = \begin{vmatrix} \frac{\partial y_0}{\partial z} & \frac{\partial y_0}{\partial x} \\ \frac{\partial z_0}{\partial z} & \frac{\partial z_0}{\partial x} \end{vmatrix}, \quad N = \begin{vmatrix} \frac{\partial y_0}{\partial x} & \frac{\partial y_0}{\partial y} \\ \frac{\partial z_0}{\partial x} & \frac{\partial z_0}{\partial y} \end{vmatrix}, \qquad (33)$$

and put

$$Q = L \cdot \frac{\partial x_0}{\partial x} + M \cdot \frac{\partial x_0}{\partial y} + N \cdot \frac{\partial x_0}{\partial z},$$

$$R^2 = L^2 + M^2 + N^2,$$

we may write the results of differentiating all the equations of (31) with respect to x_0, y_0, z_0, in the form

$$\frac{\partial x}{\partial x_0} = \frac{L}{Q}, \frac{\partial y}{\partial x_0} = \frac{M}{Q}, \frac{\partial z}{\partial x_0} = \frac{N}{Q}, \tag{34}$$

so that

$$\frac{\partial p}{\partial x_0} = \frac{\frac{L}{R} \cdot \frac{\partial p}{\partial x} + \frac{M}{R} \cdot \frac{\partial p}{\partial y} + \frac{N}{R} \cdot \frac{\partial p}{\partial z}}{\frac{L}{R} \cdot \frac{\partial x_0}{\partial x} + \frac{M}{R} \cdot \frac{\partial x_0}{\partial y} + \frac{N}{R} \cdot \frac{\partial x_0}{\partial z}}, \tag{35}$$

and this is evidently equal to (9), the ratio of the directional derivatives of p and x_0 taken in the direction (s) in the (x, y, z) space in which both y_0 and z_0 are constant. If (s, p), (s, x) represent the angles between s and the directions of the gradient vectors of p and x respectively,

$$\frac{\partial p}{\partial x_0} = \frac{h_p \cdot \cos(s, p)}{h_{x0} \cdot \cos(s, x_0)}. \tag{36}$$

It is convenient, therefore, to define the derivative of a scalar point function, u, with respect to another scalar point function, v, at any given point in any direction (s), as the ratio of the directional derivatives of u and v taken at the point in the direction s.

Derivatives of this kind which frequently appear in two-dimensional problems in Thermodynamics and in Hydrokinematics, usually involve, as has been said, a transformation from one set of coördinates to another which is not orthogonal.

XV

THE EFFECT OF LEAKAGE AT THE EDGES UPON THE TEMPERATURES WITHIN A HOMOGENEOUS LAMINA THROUGH WHICH HEAT IS BEING CONDUCTED [1]

In many of the determinations of thermal conductivity which have been made during the last few years, the so-called "wall method" has been employed. That is, one face of a plate or wall of the material to be experimented upon has been kept at one constant temperature for a long time while the opposite face has been maintained at another constant temperature, and the quantity of heat per square centimeter of either face, which under these circumstances has passed per second from one face to the other, has been measured in some convenient way.

In practice such a plate is of limited dimensions, and although it is easy to insure that the temperatures of the faces shall be nearly uniform, it is comparatively difficult to maintain a steady gradient from face to face at the edges so that the heat flow within the slab shall be the same as if the faces were infinite in extent. If, however, the faces of the specimen to be used are small enough, it is possible to prevent almost entirely the escape of heat at the edges by surrounding the periphery by an arrangement like a Dewar flask. This is impracticable when for any reason the plate has to be large, and in this case it is necessary to make the thickness of the wall so small compared with the dimensions of the faces that the lines of flow of heat from face to face in the central portion of the slab shall not be appreciably distorted by loss of heat through the edges of the wall.

Some time ago, in an attempt to obtain an accurate average value of the conductivity of a given stratum in a certain deep mine, I had occasion to apply the wall method to some blocks of stone which were not perfectly homogeneous, and in order to represent the material fairly it seemed best to use a slab eight centimeters thick for each determination. The slabs were square and the edges were covered with

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlv, no. 13, April, 1910.

lagging to make the loss of heat through them as small as possible. Under these circumstances there was a very rough approximation to a uniform temperature gradient from the warm face to the cold one, at each edge, but it was difficult to measure the edge temperatures accurately and the areas of the faces were therefore made so large that the temperatures of points on the axis of the slab (that is, the line which joins the centres of the faces) would surely be the same within one one-hundredth of a degree of the centigrade scale, in the final state, whether the whole of each edge was kept at the temperature of the warmer face or at the temperature of the colder face.

In anticipation of some further work of the same kind, I have been led to compute the final axial temperatures in a square slab ($a \times a \times c$) of thickness c, when one face is kept at temperature T_0 while the other face and all the edges are kept at the lower temperature T_1. The work is straightforward enough, but the computation when the slab is relatively broad is very laborious, and in view of the practical importance of the wall method in determinations of the conductivities of poor conductors of heat, it seems well to record some of the results.

The problem just stated is solved ($T_1 - WT_1 + WT_0$) when one has found [1] a solution (W) of the equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \tag{1}$$

which is equal to unity when $z = 0$, and to zero when $z = c$ for all positive values of x and y not greater than a; and which vanishes when $x = 0$, or $y = 0$, or $x = a$, or $y = a$, for all positive values of z not greater than c.

A convenient normal solution of (1) is

$$A\,(e^{\frac{k\pi z}{a}} - e^{\frac{k\pi(2c-z)}{a}}) \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}, \tag{2}$$

where $k^2 = m^2 + n^2$, and it is evident that

$$W\,(x, y, z) = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \left(\frac{16}{\pi^2 mn \sin \mathrm{h}\,\frac{\pi kc}{a}} \cdot \sin \mathrm{h}\,\frac{\pi k(c-z)}{a} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a} \right) \tag{3}$$

where m and n are odd integers.

[1] Byerly, *Fourier's series*, etc., p. 127.

The function

$$V = 1 - W(x, y, c - z), \tag{4}$$

which satisfies (1), is equal to unity when $z = 0$, and also for all positive values of z not greater than c, when $x = 0$, or $y = 0$, or $x = a$, or $y = a$. It vanishes when $z = c$, and the function

$$U = T_1 - W(T' - T_0) - V(T_1 - T') \tag{5}$$

or

$$T' - W(x, y, z) \cdot (T' - T_0) + W(x, y, c - z) \cdot (T_1 - T') \tag{6}$$

gives the temperatures in the slab if one face is kept at the temperature T_0, the other face at T_1, and the edges at T'. In an infinite slab of thickness c, the faces of which are kept at T_0 and T_1, the temperatures are given by the expression

$$U_\infty = (T_1 - T_0)\frac{z}{c} + T_0 \tag{7}$$

so that the difference between the values of the temperature at any point in the slab in the ideal case and the real case is

$$(T_1 - T_0)\left[\frac{z}{c} - W(x, y, c - z)\right] + (T' - T_0)[W(x, y, z) + W(x, y, c - z) - 1] \tag{8}$$

The last factor of this expression has its maximum value at the middle point of the axis where $z = \frac{1}{2}c$.

The value of W for the centre of the axis of the slab is given for several different values of a in Table I. When the ratio of a to c is large, the double series which defines W converges very slowly. Thus to obtain the last number in the table more than one hundred and fifty terms of the series were needed.

Figure 1 represents the numbers of Table I graphically.

It is interesting to compare these results with similar ones for circular disks which Professor R. W. Willson and I obtained [1] several years ago.

[1] *Proceedings of the American Academy*, vol. xxxiv, No. 1, 1898.

TABLE I

a	W	a	W
$\frac{1}{4}$ c	0.014	$\frac{3}{2}$ c	31.570
$\frac{1}{2}$ c	1.176	2 c	40.708
$\frac{2}{3}$ c	5.720	3 c	47.556
$\frac{4}{5}$ c	9.833	5 c	49.905
c	16.666		

TABLE II

Final Axial Temperatures in a Homogeneous Disk of Diameter d and Thickness c, when one Face ($z = 0$) is kept at 100° C., the other Face ($z = c$) at 0° C., and the Edge at the Uniform Temperature $\bar{\theta}$.

d/c	z/c	$\bar{\theta}=0°$	$\bar{\theta}=100°$	$\bar{\theta}=50°$
$\frac{1}{2}$	$\frac{1}{4}$	14.05	99.88	56.95
$\frac{1}{2}$	$\frac{1}{2}$	1.30	98.70	50.00
$\frac{1}{2}$	$\frac{3}{4}$	0.12	85.95	43.03
1	$\frac{1}{4}$	42.32	96.07	69.20
1	$\frac{1}{2}$	13.93	86.07	50.00
1	$\frac{3}{4}$	3.95	57.68	30.80
$\frac{3}{2}$	$\frac{1}{4}$	58.15	88.83	73.49
$\frac{3}{2}$	$\frac{1}{2}$	28.54	71.46	50.00
$\frac{3}{2}$	$\frac{3}{4}$	11.17	41.85	26.51
2	$\frac{1}{4}$	66.41	82.86	74.63
2	$\frac{1}{2}$	38.39	61.61	50.00
2	$\frac{3}{4}$	17.14	33.59	25.36
3	$\frac{1}{4}$	72.84	77.12	74.98
3	$\frac{1}{2}$	46.98	53.02	50.00
3	$\frac{3}{4}$	22.88	27.16	25.02
4	$\frac{1}{4}$	74.48	75.51	74.99
4	$\frac{1}{2}$	49.27	50.73	50.00
4	$\frac{3}{4}$	24.49	25.52	25.01
6	$\frac{1}{4}$	74.97	75.03	75.00
6	$\frac{1}{2}$	49.96	50.04	50.00
6	$\frac{3}{4}$	24.97	25.03	25.00
10	$\frac{1}{4}$	75.00	75.00	75.00
10	$\frac{1}{2}$	50.00	50.00	50.00
10	$\frac{3}{4}$	25.00	25.00	25.00

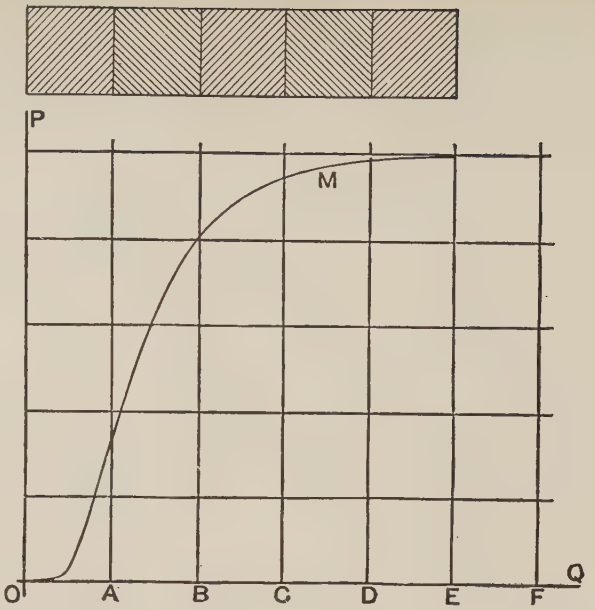


FIGURE 1. — The ordinates of the curve show the temperatures, for different values of a, of a point Q in the centre of the axis (OS) of a square slab $(a \times a \times c)$ of given thickness c, when one face $(a \times a)$ is kept at the temperature 100° while the other face and the edges are kept at 0°. The horizontal unit is c, and it appears that when $a = 5\,c$, the temperature (49.9°+) of Q differs only slightly from the temperature (50°) which it would have if a were infinite. The shaded area above indicates the section of the slab for different values of a.

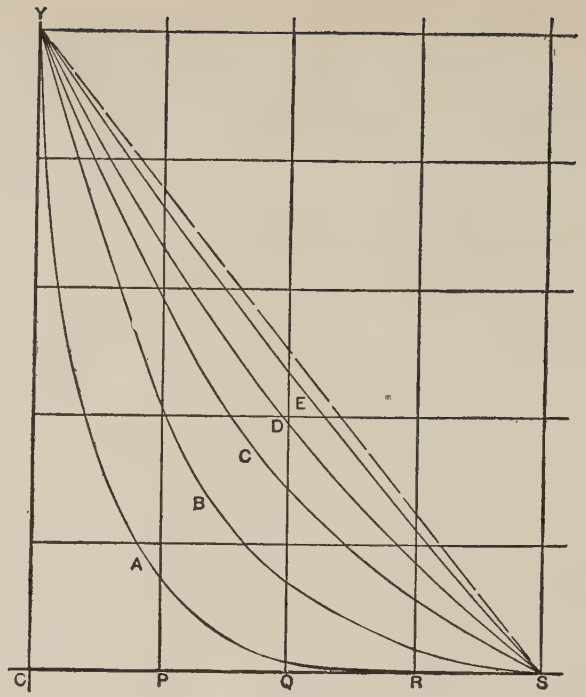


FIGURE 2. — The curves show the final temperatures on the axis (OS) of a circular disk of given thickness (c) and of diameter d, when one face is kept at the temperature 100° and the other face and the rim at 0°. In A, B, C, D, and E, the diameter has the values $\frac{1}{2}\,c$, c, $\frac{3}{2}\,c$, $2\,c$, $3\,c$, respectively.

XVI

THE MAGNITUDE OF AN ERROR WHICH SOMETIMES AFFECTS THE RESULTS OF MAGNETIC TESTS UPON IRON AND STEEL RINGS [1]

THE theory of the magnetic properties of a homogeneous ring of iron or steel uniformly wound about by turns of insulated wire through which a steady current of electricity can be made to pass, was first investigated by Kirchhoff, who showed that the intensity of the magnetic field in the metal which thus forms the core of a ring solenoid, must be inversely proportional to the distance from the axis of revolution of the ring. He computed the mean value of the field in a ring of rectangular cross section,[2] and pointed out the advantages which rings offer for measurements of the magnetic permeabilities of the metals of which they are made. The next year, Stoletow, working under the advice of Kirchhoff, took up the subject practically in the Physical Laboratory of the University of Heidelberg and in 1872 published the results of a long series of experiments upon a ring forged from a wrought iron rod. In 1873 appeared an account of the important work of Rowland, begun three years before, on rings (toroids) of circular cross section, made of various kinds of iron and steel, and since that time countless measurements of permeability have been made by many observers [3] upon iron and steel rings; and when these rings have been turned out of masses of solid metal, and not forged up and welded from bars, the results have usually been satisfactory.

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlvi, no. 3, September, 1910.

[2] Gustav Kirchhoff, *Pogg. Ann.*, Ergzbd., vol. v, 1, 1870. *Gesammelte Abhandlungen*, 223; A. Stoletow, *Pogg. Ann.*, vol. cxlvi, 442, 1872; H. Rowland, *Phil. Mag.*, vol. xlvi, 140, 1873; vol. xlviii, 321, 1874; C. Bauer, *Wied. Ann.*, vol. xi, 349, 1880.

[3] J. A. Ewing, *Magnetic Induction in Iron and other metals;* G. vom Hofe, *Wied. Ann.*, vol. xxxvii, 482, 1889; H. Lehmann, *Wied. Ann.*, vol. xlviii, 406 1893; A. von Ettingshausen, *Wied. Ann.*, vol. viii, 554, 1879; H. du Bois, *Magnetische Kreise*, 110 *et seq.* Berlin, 1894; L. Boltzmann, *Anz. d. Wiener Akademie*, 203, 1878; J. Sauter, *Wied. Ann.*, vol. lxii, 85, 1897; L. Mues, *Inaug. Diss. Greifswald*, 1893; I. Schuetz, *Journal f. Mathematik*, vol. cxiii, 161, 1894; Carl Neumann, *Ueber Ring Potentiale*, Halle, 1864.

The value of the mean intensity of the magnetic force within the mass of a ring of circular cross section was given without proof by Bauer in 1880; a proof was printed by Lehmann in 1893, and an interesting diagram based on the formulas of Kirchhoff and Bauer, and showing the ratio of the mean magnetizing force to the value of the force at the mean radius for rings and toroids of different relative dimensions, was given by Morton [1] in the Bulletin of the Bureau of Standards for February, 1909.

In determining the permeability of an iron ring it is usual to demagnetize the metal as thoroughly as possible at the outset, and then, either by the "Method of Ascending Reversals" or the "Step-by-step Method" to determine for each of a number of values of the magnetomotive force, the whole flux of magnetic induction through the ring. The ratio (B') of this flux to the area of the cross section of the ring is then plotted against the mean value of the magnetic force in the metal to get an HB diagram for the given magnetic journey of the iron. It is clear, however, as the earliest workers in this field saw, that the process here described is only approximately exact, for the induction often has very different values at the points of the ring nearest the axis of revolution and at those farthest away from it. Indeed, in a ring of soft iron of the dimensions of the specimens employed in a well-known form of commercial testing apparatus used in Europe, the value of B at points on the inner edge of the ring when the average value of the force in the metal is unity, may be as high as 2,000, while the value at points at the outer surface is only 700. In this case there is a considerable difference between the average value (B') of the flux in the metal and the real value (B'') of B at points of the ring where H has the average value. For relatively slender rings and fairly high excitations the discrepancy is not so great, and various attempts have been made to estimate its amount beforehand for materials of different kinds. It sometimes happens, however, that one has at command only a small piece of the iron to be tested, and it becomes necessary to make the measurements upon a relatively

[1] J. A. Moellinger, *Elect. Zs.*, vol. xxii, 379, 1901; B. Soschinski, *Elect. Zs.*, vol. xxiv, 292, 1903; J. W. Esterline, *Proc. Am. Soc. for Testing Materials*, vol. iii, 288, 1903; R. Richter, *Elect. Zs.*, vol. xxiv, 710, 1903; R. Edler, *Mitt. Techn. Gewerbe-Museums*, Vienna, vol. xvi, 67, 1906; M. G. Lloyd, *Bulletin of the Bureau of Standards*, February, 1909.

stout ring not much larger than a finger ring, as Dr. A. Campbell of the National Physical Laboratory, Teddington, Middlesex, England, has so successfully done. If in such a case great accuracy is required, the work has to be carried out with considerable care and some attention has to be paid to the fact that there is a real, if usually small, difference between the value of B corresponding to the mean H, and the mean value of B.

I have had occasion of late to determine the permeability of a small ring of extremely pure soft iron, and have found it helpful to compute by the aid of accurate HB diagrams, previously made for two or three different kinds of iron and steel in the form of long rods, what the discrepancy $(B' - B'')$ would be for these materials at different excitations, if they were made into rings of the dimensions of the one I was compelled to use. This paper gives some results which seem instructive, for a very soft kind of Norway wrought iron and for a specimen of Bessemer steel fairly typical of what one meets with in practice.

The straight rods used were magnetized and demagnetized in a uniform solenoid about five meters long, consisting of 20904 turns of well-insulated wire wound on a stout, solid-drawn brass tube through which a stream of tap water could be kept running about the rod to prevent any sensible rise of temperature. The axis of the solenoid was horizontal and perpendicular to the meridian. The flux of induction in the rods was measured by means of a test coil of fine wire wound on the rod at its centre. This coil was protected by rubber tape and its leads were insulated from the water by rubber tubes of fine bore slipped over them. The ballistic galvanometer employed had a period so long [1] that no detectable error was introduced into the readings by the fact that a measurable time was needed to make the magnetic changes incident to a reversal of current in the solenoid. The rods were demagnetized by means of a long series of currents in the solenoid, alternating in direction and gradually decreasing in intensity; and the fact that this process was successful showed that the rods were practically homogeneous throughout. The rods were so long that the corrections for the ends, as given by Du Bois or by Shuddemagen,[2] were very small.

[1] Peirce, *Proceedings of the American Academy*, vol. xliv, 283, 1909.
[2] C. L. B. Shuddemagen, *ibid.*, vol. xliii, 183, 1907.

Tables I and II give the results of determinations [1] of corresponding values of H and B made by the method of ascending reversals by Mr. John Coulson and myself. A number of diagrams were obtained for each rod to make sure that the rather elaborate apparatus for demagnetizing the specimens was effective and that the metal was practically homogeneous throughout, and although the larger values of B are given in the tables rather more exactly than the observations warrant,

TABLE I

Annealed Norway Iron Rod 1.25 cms. in Diameter, Magnetized in a Uniform Solenoid about Five Meters Long. Results Obtained by the Method of Reversals

H	B	H	B	H	B
0	0	4.0	11280	16.	15700
0.2	90	4.5	11980	18.	15900
0.4	260	5.0	12560	20.	16040
0.5	395	5.5	13000	25.	16320
0.6	570	6.0	13400	30.	16520
0.8	1125	6.5	13700	35.	16740
1.0	2150	7.0	13900	40.	16920
1.2	3160	7.5	14100	45.	17100
1.4	4140	8.0	14300	50.	17220
1.5	4600	8.5	14490	60.	17450
1.6	5200	9.0	14660	70.	17630
1.8	5825	9.5	14800	80.	17820
2.0	6600	10.	14940	90.	18020
2.5	8240	11.	15100	100.	18210
3.0	9480	12.	15360	105.	18300
3.5	10460	14.	15540		

the slow-moving ballistic galvanometers employed permitted of very accurate measurements of the flux changes in the testing coil.

After a good HB diagram, accurately drawn on a large scale, has been obtained for a given kind of iron or steel, it is possible to find out how nearly the mean value of the magnetic induction in a given ring made of this material would differ from the real induction corresponding to the mean value of the field in the metal, for any given excitation. Suppose, for example, that the ring is to be a toroid and that the radius of the circular cross section is to be a, while the centre of the section is distant c cms. from the axis, OY, of revolution of the ring. Suppose that the excitation is to be such as to make the value of H, at points distant c from OY, H_c, then the value of H at a point P (Figure 1) is

[1] Peirce, *American Journal of Science*, vol. xxvii, 273, 1909.

$H_c\, c/(\mathrm{OP})$. Let the numerical value of this quantity be computed for say $n + 1$ points evenly dividing the space WV, and let the numerical values of B corresponding to these values of H be read with the help of a lens from the HB diagram. Let P represent one of the points of division, let y represent the product of the value of B corresponding to the value of H at P, and the width, ST, of the ring at P, and let a curve be drawn with the y's as ordinates and the OP's as abscissas. The ratio of the area under this curve — obtained by the help of a

TABLE II

Rod of Bessemer Steel 1.25 cm. in Diameter, Magnetized in a Uniform Solenoid about Five Meters Long. Results Obtained by the Method of Reversals

H	B	H	B	H	B
0	0	4.5	4830	40.	15200
0.2	40	5.0	5700	50.	15720
0.4	90	5.5	6410	60.	16120
0.5	115	6.0	7060	70.	16460
0.6	145	6.5	7650	80.	16750
0.8	210	7.0	8130	90.	17000
1.0	280	7.5	8600	100.	17220
1.2	354	8.0	9040	120.	17640
1.4	426	8.5	9420	160.	18240
1.5	465	9.0	9760	200.	18800
1.6	505	9.5	10070	240.	19200
1.8	610	10.0	10390	270.	19620
2.0	730	10.5	10650	300.	19800
2.5	1180	11.0	10890	350.	20240
3.0	1935	15.	12400	400.	20660
3.5	2800	20.	13360		
4.0	3760	30.	14480		

good Amsler's planimeter — to πa^2, gives the mean value (B') of the magnetic induction in the ring. The average value of the field (H) is $\frac{2cH_c}{a^2}(c - \sqrt{c^2 - a^2})$ and the value (B'') of the induction corresponding to this value of H can be found from the HB diagram.

An illustration may help to make the details of the process more intelligible. Consider a toroid of the Norway iron, the circular cross section of which has a radius of one centimeter and the mean radius (c) of which is 7 centimeters. If the excitation is to be such that the value of the magnetic field at C is unity, the values of H at points distant 6, 6.1, 6.2, 6.4, 6.6, 6.8, 7.0, 7.2, 7.4, 7.6, 7.8, 8 cms. respectively

from the axis of revolution of the ring, are 1.167, 1.147, 1.129, 1.094, 1.061, 1.030, 1.000, 0.972, 0.946, 0.921, 0.897, 0.875, and the values of B which correspond to these as determined from the HB diagram for the iron, are 2995, 2895, 2810, 2635, 2460, 2310, 2150, 2005, 1870, 1745, 1620, 1505. If 2895 be multiplied by the thickness of the ring at a distance of 6.1 cms. from the axis (OY), 2810 by the thickness at a distance of 6.2 cms. from OY, etc., a curve of the form KNQ shown in Figure 1 will be obtained. This curve was actually laid down on a

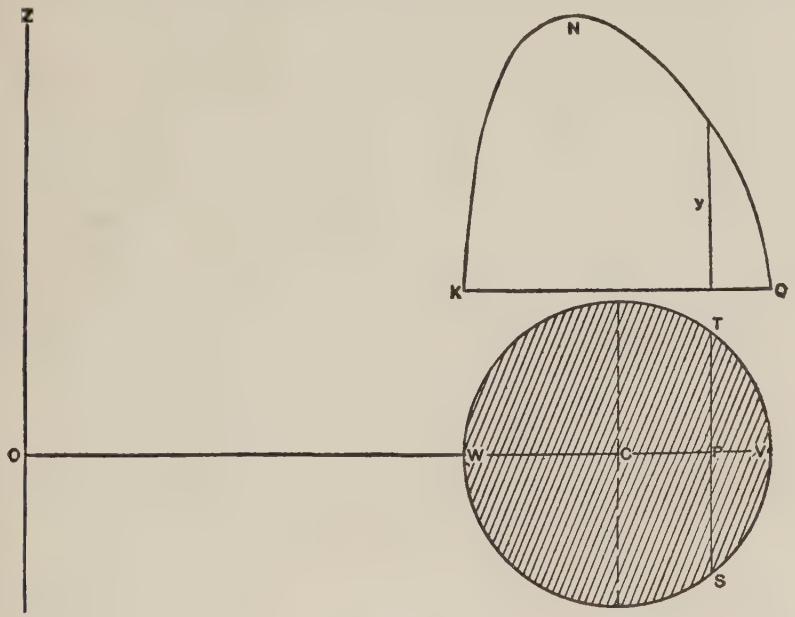


FIGURE 1

large scale by the help of a needle point on a sheet of good coördinate paper, and the area under it was determined to be 6816, though the last significant figure is not determined. This divided by πa^2 gives 2170 as the mean value (B') of B in the ring. The mean value of H in the ring is 1.0052 and the value (B'') of B which corresponds to this is 2176. Although these results have been obtained with great care, they cannot of course be assumed to be quite correct; but it appears to be true that the error in this case is not very large.

The corresponding process in the case of a ring with rectangular cross section is much simpler and the results are more trustworthy,

for the ring has a uniform thickness and the curve which bounds the nearly trapezoidal area to be measured often has so slight a curvature that the application of some form of Simpson's Rule may be made to yield a result much more accurate than a planimeter can be expected to furnish.

In the case of such a ring, as appears from the last two columns of Tables III, IV, and V, B' is usually a trifle larger than B'', for very

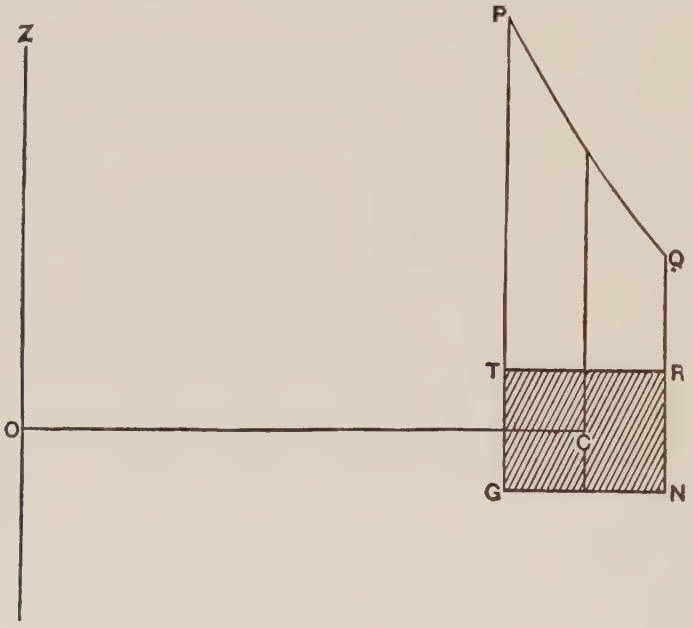


FIGURE 2

small values of the mean H in the iron, but is equal to it for a single somewhat larger value. Then, with increasing values of H, B' is a trifle smaller than B''; but the ratio B'/B'' soon approaches unity from the under side, and, for high excitations, is sensibly equal to one. It is evident, however, that the form of B'/B'' as a function of the average value of H in the ring must depend upon the dimensions of the latter as well as upon the magnetic properties of the material of which the ring is made.

If a ring of rectangular cross section, of the same inner and outer diameters as the toroid just described, be made of the Norway iron,

and if the excitation be made such that the average value of the magnetic field in the metal at the centre (C) of the cross section is unity, the values of H and B already found may be used to draw the curve PQ, Figure 2. The area under this curve as computed by Simpson's

TABLE III

RING OF THE PURE ANNEALED NORWAY IRON. (RECTANGULAR CROSS SECTION. INNER RADIUS, 2a; OUTER RADIUS, 3a)

H_0	H_m	H_1	B_0	B_m	B_1	H'	B'	B''
0.5	0.4	0.33	400	280	190	0.405	267	255
1.0	0.8	0.67	2150	1140	710	0.811	1235	1195
1.5	1.2	1.00	4600	3155	2150	1.216	3435	3230
2.0	1.6	1.33	6600	5020	3835	1.622	5090	5130
3.0	2.4	2.00	9480	7950	6600	2.433	7970	8040
4.0	3.2	2.67	11280	9870	8660	3.244	9910	9950
6.0	4.8	4.00	13400	12350	11280	4.866	12340	12410
8.0	6.4	5.33	14300	13630	12880	6.488	13610	13680
10.0	8.0	6.67	14940	14300	13760	8.110	14310	14330

H_0 and B_0 are the values of the magnetic force and of the induction at the inner surface of the ring; H_1, B_1 and H_m, B_m the values of the same quantities at the outer surface of the ring and at the mean radius, respectively. H' is the mean value of the magnetic field in the steel. B' is the mean value of the induction in the ring as obtained by mechanical integration from a diagram of ascending reversals for the steel, and B'' is the value of the induction corresponding to H' as shown by the same diagram. The table shows the error made by using B'/H' instead of the exact value B''/H' for the permeability corresponding to H'.

TABLE IV

THINNER RING OF THE ANNEALED NORWAY IRON. (RECTANGULAR CROSS SECTION. INNER RADIUS, 4a; OUTER RADIUS, 5a)

H_0	H_m	H_1	B_0/B_2	B	H'	B'	B''
0.62	0.555	0.50	1.59	405	0.558	502	490
0.94	0.833	0.75	1.90	1275	0.837	1325	1310
1.25	1.111	1.00	1.58	2702	1.116	2728	2740
2.50	2.222	2.00	1.25	7365	2.231	7386	7404
3.75	3.333	3.00	1.15	10130	3.347	10140	10150
6.25	5.555	5.00	1.08	13060	5.578	13070	13080
10.00	8.888	8.00	1.04	14600	8.924	14610	14620

Rule appears to be 2185, and the value of B corresponding to the average value 1.0069 of H is also 2185, so for these dimensions and for this particular excitation, the error represented by $B' - B''$ seems non-existent.

For an excitation great enough to make the value of H at the mean radius 2, a process similar to that just described shows that B' would be 6630 and B'', 6650, but if H_c were made 5, the value of B' would

be 12560 and B'', 12590. The difference in this instance is less than one quarter of one per cent of either quantity and lies within the limits of error of most magnetic measurements made upon ring specimens. For work that must be very accurate, rings much thinner than this one — in which the ratio of the outer radius to the inner radius

TABLE V

RING OF THE BESSEMER STEEL. (RECTANGULAR CROSS SECTION. INNER RADIUS, 2A; OUTER RADIUS, 2A)

H_0	H_m	H_1	B_0	B_m	B_1	B_0/B_1	H'	B'	B''
1.0	0.8	0.67	280	201	170	1.65	0.811	209	205
2.0	1.6	1.33	725	502	385	1.88	1.622	521	510
3.0	2.4	2.00	1920	1060	725	2.65	2.433	1150	1100
4.0	3.2	2.67	3760	2260	1360	2.77	3.244	2365	2330
6.0	4.8	4.00	7060	5350	3760	1.88	4.866	5370	5470
8.0	6.4	5.33	9030	7530	6190	1.46	6.488	7550	7630
10.	8.0	6.67	10490	9030	7800	1.35	8.110	9060	9130
20.	16.	13.33	13360	12660	11830	1.06	16.219	12630	12710

is 4/3 — are usually employed, and the error is then practically negligible for almost all excitations.

If magnetic measurements are to be made upon rings of the dimensions sometimes used in practical permeameters, the errors arising from the difference between B' and B'' become relatively important as appears from Tables III and V, where the results for two practical cases are given. A comparison of Tables III and IV will show how fast the error decreases when the ring is made thinner.

My thanks are due to the Trustees of the Bache Fund of the National Academy of Sciences for the loan of apparatus.

XVII

THE RESISTIVITY OF HARDENED CAST IRON AS A MEASURE OF ITS TEMPER AND OF ITS FITNESS FOR USE IN PERMANENT MAGNETS [1]

It has long been known that the specific electrical resistance of a piece of soft tool steel is materially less than that of the same piece after it has been hardened, and that the relaxing of the temper of any piece of steel or iron makes the specific resistance less; but the first systematic study of this phenomenon was made by Messrs. Barus and Strouhal whose work is summarized in Bulletin 14 of the U. S. Geological Survey.[2]

In one experiment which these gentlemen made upon rods of "English Silver" steel, 0.15 cm. in diameter and all originally glass-hard, different pieces were tempered by heating them to different fairly high temperatures, as indicated by the oxide tints on their surfaces and were then cooled. When the specimens thus treated were tested it appeared that the harder the temper, the higher was the specific resistance (s) referred to the centimeter cube, and the lower the temperature coefficient (a) of the specific resistance. In the case of a certain glass-hard rod, s, in microhms, was 45 and a was 0.0016; while in a thoroughly annealed rod of the same lot, s was 16 and a about 0.0040. From these and similar experiments, Barus and Strouhal made out a table connecting s and a which they subsequently found to fit other kinds of steel pretty well. Some of their results are given in Table I.

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlvi, no. 8, October, 1910.

[2] Wiedemann, *Lehre von der Elektricität*, vol. i, 502; Benoit, *Comptes Rendus*, vol. lxxvi, 342, 1873; Matthiessen and Vogt, *Pogg. Ann.*, vol. cxxii, 10, 1864; Auerbach, *Wied. Ann.*, vol. v, 1878; *Wied. Ann.*, vol. viii, 479, 1879; Callendar, *Phil. Trans.* 1887; Strouhal and Barus, *Wied. Ann.*, vol. xi, 976, 1880; Barus, *Phil. Mag.*, vol. viii, 341, 1879; Chernoff, *Vortrag gehalten in der Russischen Technischen Gesellschaft*, 1868; Jarolimek, *Dingler's Journal*, vol. ccxxi, 436, 1876; Jarolimek and Ackermann, *Zeitschrift für das Chemische Grossgewerbe*, 1880; Percy-Wedding, *Eisenhüttenkunde*, vol. ii, 130, 1864; Karsten, *Karsten und von Dechen's Archiv*, vol. xxv, 223, 1853; Barus and Strouhal, *Bulletin o the United States Geological Survey*, no. 14; Caron, *Comptes Rendus*, vol. lvi, 43, 1863; Barus, *Phys. Rev.*, vol. xxx, 348, 1910.

If corresponding values of s and a be used as coördinates, a fairly smooth curve results, and the mean values of 79 for s and 0.0013 for a which Barus and Strouhal got for three pieces of cast iron which they tested, yield a point which seems to lie closely enough upon the prolongation of this curve. It appears also that the values of s and a which Matthiessen, Vogt, and Benoit obtained for different kinds of

TABLE I

s	a	s	a
10	0.0050	30	0.0024
12	46	32	22
14	42	34	21
16	39	36	20
18	36	38	19
20	34	40	18
22	32	42	17
24	29	46	16
26	27	50	15
28	26	60	14

wrought iron agree numerically with the values for steel; and some persons have thought that it is possible to determine the position of any piece of iron or steel in the scale of mechanical hardness, without any knowledge of the percentage of combined carbon, by finding s alone.

For bar magnets or for simple bent magnets, fine tool steel, or better, some of the kinds of special magnet steel, serve very well, but if a permanent magnet is required of such a shape that the steel has to be heated red hot a number of times during the process of forging and before it is made glass-hard, irregular temper thus introduced into the material often shows itself in the presence of irregular magnetization when the magnet is finally charged, and this sometimes makes the magnet worthless. For this and other reasons, some makers of electrical instruments are now using chilled cast iron for such magnets, and these have usually proved to be satisfactory. They are cheap, they can be made quite as strong as tool steel magnets of the same dimensions, they are very permanent after they have once been aged, and the temperature coefficients of their magnetism are almost always much smaller than those of forged steel magnets. Cast iron for permanent magnets must, however, be really hard, and, unfortunately,

mechanical tests of the hardness of this metal are often deceptive; it seems desirable, therefore, to inquire whether the electric resistivity of a piece of chilled cast iron is a criterion of its temper.

This paper gives briefly a few of the results of a large number of observations made originally with the object of testing the relative efficiencies of different methods of hardening cast iron for magnets, in use in the Jefferson Laboratory. The details of this work have mainly a local interest and are not enumerated here, but some general facts may be useful to persons who have to make such magnets for themselves.

Each of the test pieces was a rod about 30 cms. long and a little less than 0.6 cm. in diameter. These were all milled down from stouter pieces about 1.5 cms. in diameter which were usually cast in sets of a dozen from a grid pattern to insure that they should be of the same kind of iron. Different specimens from the same grid, however, often showed different resistivities before they were annealed and occasionally one or two pieces from a grid would differ sensibly from the other pieces after all had been softened with great care. These differences are to be expected, as Karsten showed long ago, for the outer layers of a mass of chilled cast iron sometimes contain a greater proportion of combined carbon than the inner layers in which most of the carbon may be free, and an unequal chilling of a grid in the mould would naturally make the material slightly different in different parts. It is easy in practice to avoid abnormal specimens. All the test pieces were prepared, annealed, and hardened by Mr. George W. Thompson, the mechanician of the Jefferson Laboratory, whose experience in treating cast iron extends over many years.

The measurements of the specific resistances of the rods (usually three for each specimen) were mostly made with the help of a standard Kelvin Double Bridge, but in a few cases the test piece was connected in series with a standard manganin resistance bar and a constant storage battery, and the small potential drop across a measured length of the rod was compared with the corresponding drop across the standard. Three commutators were used with this apparatus so that the effects of disturbing electromotive forces at the contacts might be avoided. The ultimate standard was Wolff No. 2718 furnished with the certificate of the Reichsanstalt.

In the determinations of the temperature coefficients of resistivity two large tanks of water were used. One of these was approximately at room temperature. The water in the other, which was kept in constant motion by a set of four propellers run by a small motor, was heated to a constant definitely determined temperature by means of a Simplex Electric Heater attached to a 110 volt circuit and dominated through a relay by a delicate thermostat. The annealing effects of very hot water upon hard cast iron had to be avoided, but the water in the second tank was usually made uncomfortably warm for the hand.

In making cast iron magnets, it is very necessary that the iron just before it is chilled shall be much hotter than it is safe to heat ordinary tool steel in making it hard. Dr. Campbell, of the National Physical Laboratory, Teddington, Middlesex, England, finds that a temperature of 1000° C. has been sufficient for the iron he has used, but some specimens of American iron seem to work best at a slightly higher temperature, just below the melting point. If a massive piece of cast iron weighing, say, fifty pounds be heated thus hot and then chilled in a proper bath, the material, as magnetic tests can be made to show, becomes hard throughout, whereas it is practically impossible to make a similar piece of tool steel glass-hard inside. The experiments of Chernoff upon a certain kind of steel, made more than forty years ago, showed that if the temperature from which the steel was chilled was made higher and higher, from, say, 400° C., the hardening effect was almost inappreciable until a cherry red was reached, when suddenly the chilled specimen was found to be glass-hard. It is not very surprising, therefore, that cast iron shows very little temper when chilled from a temperature of 800° C. or 900° C., but may easily be made glass-hard if its temperature just before the chilling is high enough, say 1050° C. for some kinds.

The rods were heated for the hardening, under a compressor blast, in a special gas furnace made for the purpose by Messrs. J. Connors and J. Coulson, and most of them were placed inside an iron tube to protect them from direct exposure to the flames. In annealing the rods they were packed in iron filings inside an iron tube closed at the ends by screw caps and heated thoroughly to a white heat for possibly 30 minutes before the tube was packed in ashes for many hours.

Although the work was done with the greatest care, it soon appeared that it is usually impossible, at least by this particular annealing process, to bring a piece of cast iron once made glass-hard back to as low a resistivity as it originally had, and if the piece be repeatedly hardened and annealed, its resistivity in the relaxed state increases every time the cycle is passed through. The diameter of the piece also increases perceptibly much as the cast-iron bars of a fire box grate grow longer with hard use. Two or three examples will show the complicated nature of the phenomena involved.

Two test pieces from the Broadway Iron Works, Cambridgeport, were annealed as they came from the foundry and then had resistivi-

TABLE II

CAST-IRON ROD FOUR TIMES HARDENED AND ANNEALED

H	*B*	*H*	*B*
1.13	57	7.70	964
1.40	79	9.15	1521
2.03	120	13.2	2910
3.31	222	20.6	4585
4.40	326	32.7	6030
5.75	518	42.4	6430
6.54	681		

ties 102.5 and 102.7 and a diameter of 0.574 cm. After both had been hardened, the resistivities at about 20° C. were 122.5 and 122.0, and after they had been again through the annealing furnace their resistivities were 108.7 and 107.1. The fourth time they were relaxed the specific resistances were 112.6 and 112.6, and their average diameters about 0.578 and 0.576. When they were finally hardened again, the resistivities were 136.7 and 137.8 and both diameters were 0.581. It did not seem worth while to carry the process further.

Another rod, presumably of a very different kind of iron, began with a diameter of 0.574 and after four annealings had a mean diameter of 0.578. Its resistivity in the relaxed state rose in four steps from 93.9 to 102.5; the first time it was hardened its resistivity was 112.0, the last time 116.5.

In the three cases here mentioned the specimens would cut common window glass easily the first time they were hardened; they were mechanically too soft to scratch the same glass when, having been

repeatedly hardened and relaxed, they were finally hardened so that they had a higher resistivity than at first.

Another rod from the same foundry had a resistivity of 102.0 when it was first annealed, and a resistivity of 119.8 when it was hardened for the first time. After an hour in steam at 100° this fell to 118.0, and after five hours farther steaming to 116.6. The second time it was annealed the rod had a resistivity of 106.5, and the third time of 107.2.

The temperature coefficient of the resistivity of the first rod spoken of above was 0.00102 when the rod was soft; the third rod had a temperature coefficient of 0.00094.

Cast iron which has been several times hardened and annealed is finally in its annealed state not so permeable as once-annealed soft cast iron is. Table II gives the results of tests upon a rod of resistivity 98.3 which has been four times heated white hot and chilled and then annealed.

If the process of heating and chilling a number of cast-iron rods be carried out many times in succession without proper annealing after each chilling, there does not seem to be a progressive increase in the resistivity; the results are anomalous.

Several kinds of chilling baths were used for hardening the cast iron, among them ice cold water, cold brine, sulphuric acid and water, an acid bath (X) the constitution of which is a trade secret, but which, I understand, has been much used in commercial work; mineral oil, and paraffine.

It has long been known that in the hardening of tool steel from a dull red heat, it is much more important that the fall of the temperature of the piece down to say 300° C. shall be quickly brought about than that the rest of the journey to room temperatures shall be rapid. It is not difficult to cool quickly a slender rod, but a large piece of hot metal suddenly immersed in a water bath is immediately surrounded by a layer of steam and, unless the water be very vigorously stirred as in die hardening, the metal may remain red hot for a comparatively long time. Many attempts have been made by varying the chemical nature of the bath to lessen the effect of the steam cloak, and some persons have used a bath of easily fusible metal for the first part of the chilling process (as is now the practice for some of the new high

power steels), and have completed the cooling in a water bath, the temperature of which within wide limits seems to be unimportant.

In the light of the behavior of steel, it seemed unlikely that in the hardening of cast iron from a temperature much higher than can be used with ordinary tool steel, there would be much advantage in making the hardening baths especially cold, and experience justified this assumption. Sometimes the hardening bath was chilled with ice, but usually it was used at room temperatures or even lukewarm.

For rods of the dimensions of the test pieces I used, water, brine, sulphuric acid and water, and the *X* mixture seemed almost equally effective in making the cast iron glass-hard, whether resistivity or magnetic permeability of the hardened piece was used as the criterion. For massive pieces of iron the *X* mixture, which certainly is very good,

TABLE III

Grid	s	a
I	76.5	0.00104
II	86.1	0.00106
III	89.9	0.00099
IV	94.2	0.00084

is said to work more uniformly than a water bath. Several specimens which were chilled in iced water and iced brine developed minute cracks which showed in irregularities when the rods were magnetized, but these, which were tested before the construction of the special gas furnace, may not have been uniformly heated. The oil bath was nearly as good, so far as increasing the resistivity of the specimen, as the water bath, but the hardened pieces did not seem so hard mechanically. The melted paraffine wax, at as low a temperature as would keep the wax liquid, also increased the resistivity of a specimen chilled in it, provided it had not been hardened before, quite as much as the water bath, but a piece thus hardened would not scratch glass.

Most of the pieces of American cast iron which I have tested had, when soft, resistivities referred to the centimeter cube, which at 0° C. would lie between 73 microhms and 104 microhms. These pieces when hardened for the first time had resistivities which at the same temperature lay between 80 and 126. Nine pieces of American cast iron tested when soft by Barus and Strouhal had on the average a resistivity at 20° C. of about 79.1 microhms with a temperature coefficient of

0.00120. Four grids, typical of the softer kinds of iron which I have used, gave on the average when soft at the same temperature the results which appear in Table III.

To show the effect of hardening upon the temperature coefficient of the resistivity, I may instance six specimens with three different coefficients when hard. (See Table IV.)

When a number of steel bars of the same length and cut from the same long rod are hardened and are then magnetized in the same

TABLE IV

Rod	s'	a'	s''	a''
1	94	0.00086	114	0.00070
2	97	0.00085	117	0.00070
3	79	0.00105	103	0.00086
4	75	0.00108	102	0.00086
5	88	0.00103	113	0.00091
6	86	0.00106	117	0.00091

s' and s'' are the resistivities at 20° C. in the soft and in the glass-hard states, respectively; a' and a'' are the temperature coefficients.

solenoid and aged, it frequently happens, as is well known, that the ultimate magnetic moments of the bars differ somewhat widely from one another; and the same thing is true of magnets made from cast-iron rods cut from the same grid. In Table V are given the magnetic moments (M) and the temperature coefficients of the moments (k) of eight typical bar magnets which have been tested with great care by Mr. John Coulson.

TABLE V

Grid	Rod	M	k	Chilling bath	s
A	1	1550	0.00022	Water	110
A	2	1580	22	Water	104
B	3	1720	24	Water	94
B	4	1740	26	Water	94
B	5	1560	26	H_2SO_4 and water	96
C	6	1410	27	H_2SO_4 and water	109
C	7	1500	27	H_2SO_4 and water	106
C	8	1600	27	"X" mixture	106

Mr. Coulson tested at the same time three magnets of the same dimensions as these but made of glass-hard Stubs tool steel. They had on the average a moment of about 1690 and a temperature coefficient of about 0.00095, which is more than three times as large as the corresponding value for cast iron.

After the moments of the eight cast-iron magnets had been determined, the rods were thoroughly demagnetized inside a solenoid through which a long series of currents, gradually decreasing in intensity and alternating in direction, could be sent. Then each was placed inside another solenoid and an HB diagram was found for it by the method of ascending reversals with the aid of a small test coil

TABLE VI

H	B	H	B
10	310	70	4250
20	720	80	4950
30	1290	90	5550
40	2100	100	5950
50	2870	120	6480
60	4100		

about its centre and a ballistic galvanometer of period sufficiently long for the purpose. Each rod was about 50.9 of its own diameters long and, according to the formula of Dr. Shuddemagen for the end corrections of rods of these dimensions, the actual magnetic intensity (H) inside the metal at the centre is equal to $H' - 0.00107\ B$, where H′ is the force inside the solenoid when the rod is removed. It is possible, therefore, to determine very approximately the relative

TABLE VII

Rod	H_{1000}	B_{120}
A1	26.3	6020
A2	25.9	6200
B3	26.5	6480
B4	25.9	6480
B5	25.2	6180
C6	25.9	5960
C7	26.1	6045
C8	25.8	6150

values of H and B from the observed values of H' and B, and the computation has been made by Mr. Coulson for these rods. The results of this work show that though the moments of the magnets differed so much among themselves, the permeabilities of the pieces of metal for excitations up to $H = 50$, at least, are much the same. Magnet B4, for instance, had a moment much larger than the moment of C6, but the value of H corresponding to an induction of 1000 was in each case about 25.9.

For the rod B4 the relation between H and B is indicated approximately at all events by the numbers given in Table VI.

Table VII gives under H_{1000} the value of the excitation corresponding to $B = 1000$, and under B_{120} the value of the induction corresponding to $H = 120$ for all the rods.

The specimens used were cast at different times in order that they might fairly represent the best mixtures used by the foundries from which they came, and in view of this fact the near agreement of the measurements recorded in this table is very striking. The differences are not greater than one might expect to find in a number of rods of fine polished drill rod from the same lot. For the present discussion it is of interest to notice that the permeabilities of the hard rods seem

TABLE VIII

H	B (Rod hard)	B (Rod soft)
100	6800	9650
200	8850	11160
300	10310	12460
400	11420	13550
500	12130	14400
600	12660	14980
10000	25650	28250
11000	26600	29300
12000	27500	30400
13000	28450	31300
14000	29400	32050
15000	30350	33600

not to be connected in any obvious way with the resistivities. For any single specimen of cast iron, however, it is well known that hardening usually decreases the permeability especially at comparatively low excitations, and Figure 1 [1] shows a rough kind of hysteresis diagram which I obtained some years ago for a cast-iron frame of several kilograms weight. Curve A corresponds to the soft state and B to the hardened state of the same piece of iron. At high excitations the difference is not so striking but is very real.

Table VIII gives approximately the results of some measurements made two or three years ago upon cylinders and isthmuses of a certain kind of cast iron from the Broadway Iron Foundry. It must be clearly understood, however, that this applies only to iron which has once been through the annealing and subsequent hardening.

[1] Identical with Fig. 8 on p. 48.

A repetition of the process makes the hardened iron mechanically softer. As we have seen, a piece of cast iron properly hardened for

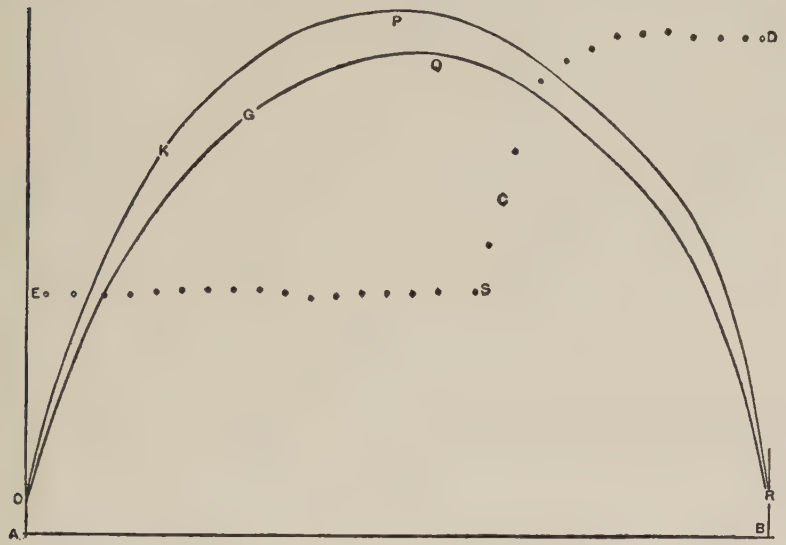


FIGURE 2

the first times makes as strong a permanent magnet as a piece of Stubs Drill Rod does, but if the cast iron be several times hardened it becomes incapable of retaining the charge given it in the solenoid and the resulting magnet is perhaps only half as strong as the steel magnet. The same phenomenon appears in the case of tool steel, though it is not very easy to harden a piece of tool steel glass-hard a number of times in succession without working it under the hammer to avoid the appearance of minute cracks in the metal.

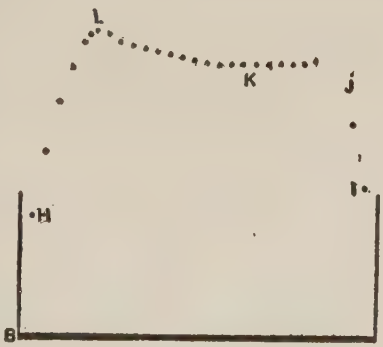


FIGURE 3

For many years small magnets made of cast iron as it comes from the founder have been used in toys and in small "magnetos," but such magnets are not nearly permanent and are not so strong at the outset

as similar magnets made of properly chilled iron. A certain annealed rod which I tested had when magnetized to saturation a moment of 605 on a certain scale, but a few minutes in boiling water reduced this to 455; when the rod had been hardened and again magnetized, its

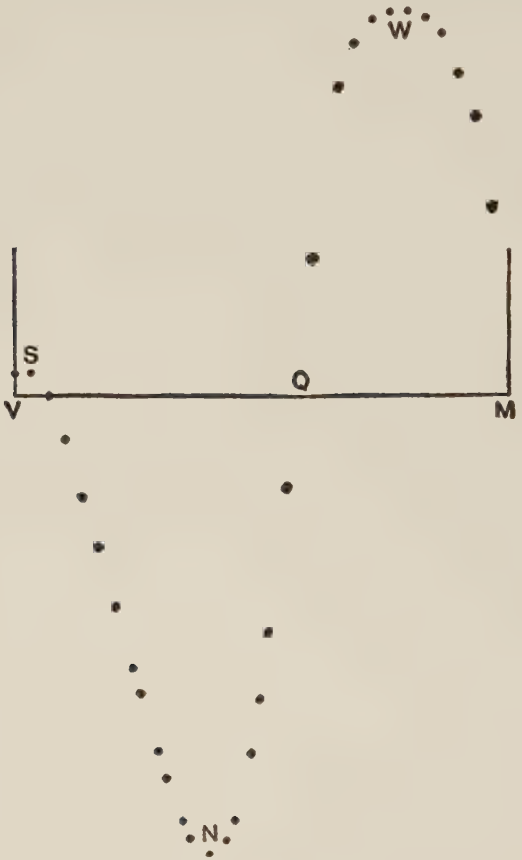


FIGURE 4

moment on the same scale as before was 831 and boiling reduced this to 740. The same magnetized castings are tested year after year in the Jefferson Laboratory, and so far as my experience goes, a properly hardened and aged magnet made of cast iron is quite permanent if it is exposed to such fields as that of the earth, and mechanical shocks do not injure them in any way, if the metal is not broken or abraded.

Although a knowledge of the resistivity of a piece of cast iron tells very little about its temper unless one knows also its resistivity in the annealed state, yet the resistivity of different portions of the same piece is a trustworthy measure of the uniformity of temper. Tried by this test, many a piece of steel which has been hardened with care proves to be far from homogeneous.

Occasionally great differences of resistivity may be found in a magnetized steel rod which yields a fairly uniform iron-filing diagram.

The curve OKPR of Figure 2 shows the induction flux (B) at different points of the axis of a rod of Crescent Polished Drill Rod 29 cm. long and 0.5 cm. diameter just after it had been magnetized to saturation in a solenoid. Curve OGQR shows the same quantity after the rod had been exposed to steam for some time. AB is the common base of these curves. The distribution is in each case nearly uniform, and the iron-filing curve seems entirely so, but the resistivity of the metal is far from uniform, as the dotted diagram ESCD shows. This was obtained by measuring the resistances of a large number of very short lengths of the rod and determining from the results values for the resistance per centimeter at about thirty points on the axis. Of course a small portion at each end could not be treated in this way, and the fact is indicated by the open dots. One end of this bar was in the soft state in which this excellent steel comes in the market; the other end had been heated red hot and chilled, so that its resistivity was quite double that of the soft end. This magnet was not so strong as a hardened magnet of this steel should be, but was otherwise normal enough.

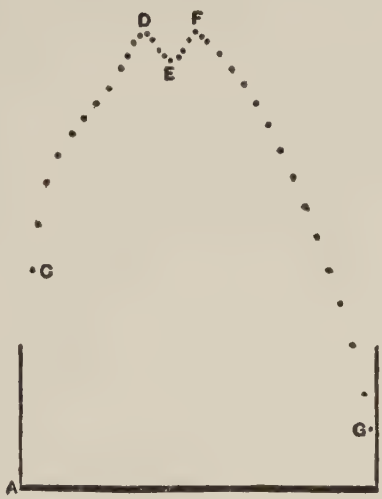


FIGURE 5

Sometimes the iron-filing diagram belonging to a bar magnet seems very irregular when the distribution of magnetism in the metal is not very abnormal. Figure A shows a filing diagram belonging to a piece of Crescent steel of the same dimensions as that just described, while

Figure 3 shows the values of B at different points in the axis. The "centre of gravity" of the magnetism is in this case not far distant from the middle of the bar. This same bar was remagnetized by rubbing a point near its centre upon one pole of a large motor, and then

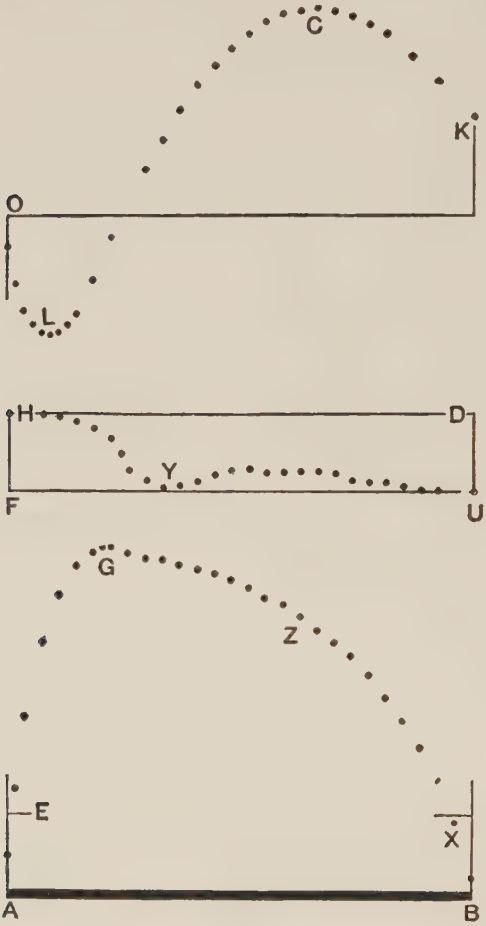


FIGURE 6

gave a filing diagram represented by Figure B. Here there are real consequent poles, and the distribution of the induction flux in the bar is shown by Figure 4. After this rod had been demagnetized as well as possible in a solenoid by the use of a series of currents alternating in direction and gradually decreasing in intensity, and then had been

magnetized again to saturation in a solenoid as before, Diagram A came back again.

Another unequally hardened steel rod of the same kind gave the filing diagram shown in Figure C, and in this case the distribution of magnetism was that indicated in Figure 5.

Figure 6 shows in the curve HYU, of which the horizontal line through E is the base, the resistivity of a rod of cast iron of the dimensions of the specimens used in this investigation. For this particular piece the resistivity at one end corresponded to the annealed state and at the other end to glass-hardness. After this rod had been magnetized in a solenoid, the distribution of magnetism in it was that

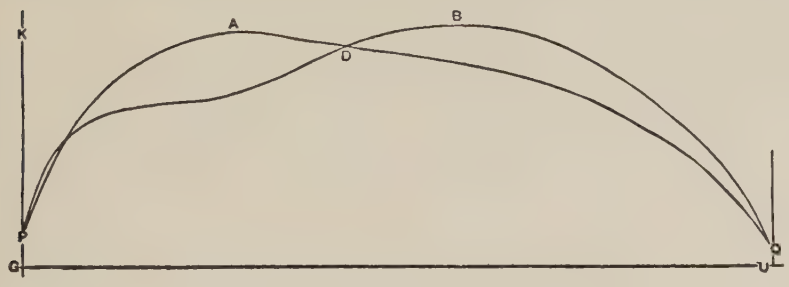


FIGURE 7

represented by the dotted curve GZX. This rod when magnetized irregularly on the motor gave the diagram LCK, but when the rod was demagnetized and again magnetized in the solenoid, the distribution GZX returned. It is interesting to notice that in the cases shown in Figures 4 and 6, the motor gave a smooth distribution of B while the solenoid gave an irregular one. When real consequent poles are present, the value of B is at its greatest, smaller than in the case of the solenoid magnetization.

Figure 7 shows in the curve PADQ the distribution of magnetism in an unequally hardened cast-iron rod when the magnetization took place in a long solenoid. Curve PDBQ shows on an exaggerated scale the distribution when the rod was magnetized between the poles of a large electromagnet. The greatest value of B was in this latter case about two-thirds the corresponding value when the solenoid was used. In all the instances I have met, the solenoid gave the greatest value of B and any other distribution gave an appreciably smaller value.

Table IX gives the resistivity at points distant n cm. from the end of the rod which corresponds to G in Figure 7. It is evident that one end of the rod is glass-hard and the other very soft.

The most common form of irregularity in a cast-iron bar magnet seems to consist, if one may judge from a filing diagram, in a simple displacement of the magnetic centre from the geometric centre towards one end of the axis. This usually corresponds to a comparatively slight difference of resistivity along the bar. This case may be

TABLE IX

n	s	n	s
3	0.000121	16	0.000097
4	122	17	94
5	121	18	94
6	121	19	92
7	120	20	92
8	117	21	92
9	115	22	93
10	111	23	94
11	104	24	94
12	097	25	94
13	097	26	93
14	097	27	93
15	097		

illustrated by a rod (K) which had once been hardened irregularly and then had been rehardened as uniformly as possible. In all such cases it is extremely difficult to get rid of the effects of careless hardening, though the irregularity may come up in a slightly different form. The next table (X) gives the resistivity of the metal, and, on an arbitrary scale, the value of B at a point distant n cm. from one end of this bar, which was 29 cm. long.

Table XI gives the resistivities and the relative values of B on the axis of a cast-iron magnet (Q) made of a rod hard in the middle and soft at the ends.

If the material used in these experiments may be considered typical of the so-called "pure cast iron" from good foundries, it appears, then, that an annealed casting may have at room temperatures a resistivity, referred to the centimeter cube, as low as 0.000073 or as high as 0.000104; that it is always possible to make the specimen glass-hard throughout by heating it to a temperature a little below the melting point and chilling it in a suitable bath; and that the process, as Barus

and Strouhal showed, is always accompanied by an increase in resistivity. This increase is sometimes only about ten per cent of the original value, though it is oftener nearly twenty-five per cent and may rise somewhat higher. Only one kind of iron that I used resisted successfully a noticeable relaxation of temper in the hardened pieces

TABLE X

n	s	B	n	s	B
0		60	15	127	551
1		227	16	126	560
2		317	17	126	568
3	121	368	18	127	572
4	118	405	19	129	578
5	119	430	20	129	580
6	118	448	21	129	578
7	120	460	22	129	573
8	121	470	23	129	565
9	121	480	24	126	535
10	122	493	25	125	505
11	123	509	26	124	453
12	123	520	27		374
13	126	532	28		260
14	126	541	29		72

TABLE XI

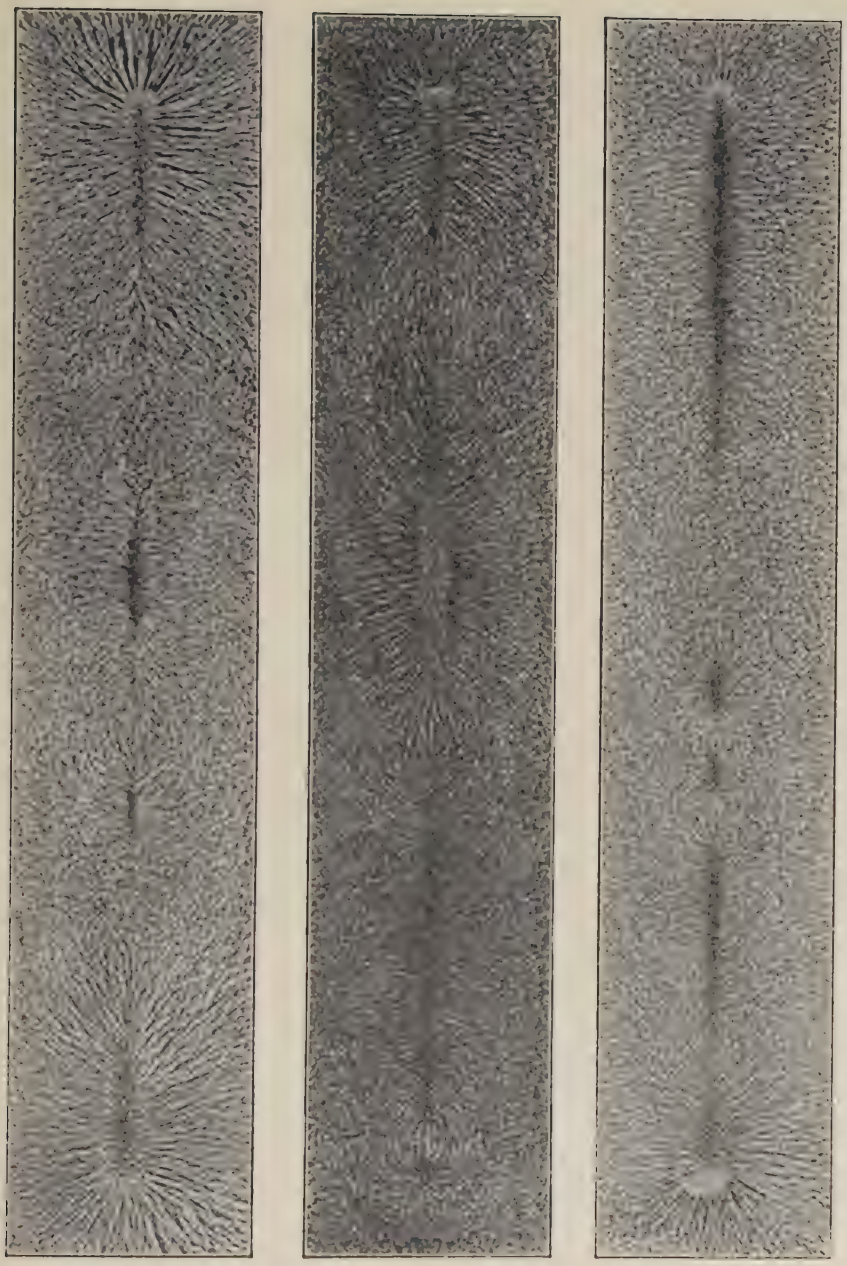
n	s	B	n	s	B
0		40	15	155	412
1		140	16	150	429
2		207	17	147	445
3	117	261	18	141	458
4	119	305	19	128	448
5	117	339	20	119	433
6	118	371	21	118	417
7	120	394	22	117	399
8	121	416	23	116	359
9	132	434	24	115	330
10	149	440	25	113	292
11	153	428	26		257
12	153	417	27		198
13	156	409	28		135
14	158	407	29		37

by prolonged boiling in water. Of two pieces of iron from the same pouring, which have equal resistivities when first annealed, that one has the higher resistivity, after both have been hardened, which has the lower magnetic permeability. Tests of mechanical hardness are difficult to make upon cast iron and often disagree with the resistivity

test. A repetition of the annealing and hardening process increases somewhat the size of a specimen and increases the resistivity for both the annealed and the chilled states, but in the hardened state the iron is never so hard mechanically as at the first hardening, and the bar loses in great measure its magnetic retentiveness, as do most kinds of tool steel which have been through the same experience. Many kinds of chilling liquids serve to make cast iron glass-hard, but for massive pieces cold water seems not to give such uniform results as the acid bath used by some professional hardeners. The temperature coefficient (a) of the resistivity of every one of my specimens was decreased by the hardening, though this does not seem to have been the case for the special cast iron used by Barus and Strouhal, which had a larger coefficient (120) than any I used. The coefficient a is not always smallest in that one of a number of specimens of cast iron which has the largest resistivity.

Castings from different sources often show when glass-hard a very close agreement in magnetic permeability, though their resistivities and the temperature coefficients of the resistivities may differ widely. The temperature coefficient of the magnetic moment of a cast-iron bar magnet is usually not more than one third as large as that of a similar magnet made of tool steel.

A uniformly hardened cast-iron or steel rod may have been irregularly magnetized, but if it be thoroughly demagnetized and then carefully remagnetized in a solenoid, its magnetism will become regular. Only irregular hardening seems to lead to persistently irregular magnetization in the case of a bar magnet, though the use nowadays of electromagnetic crane lifters sometimes magnetizes iron and steel rods in a manner which is difficult to deal with in the laboratory. Even an irregularly hardened slender rod may usually be demagnetized well enough for all practical purposes in a solenoid which carries currents alternating in direction and gradually decreasing in intensity, but large thick pieces are very tenacious of charges once given to them. The shield of a certain Rubens Panzer galvanometer in use in the Jefferson Laboratory was twice heated white hot and was kept hot for some time in a vain attempt to get rid of a slight magnetization. The resistivity of different portions of a casting gives trustworthy information about the uniformity of the hardening. Occa-



A B C

sionally, as in a case cited above, an irregularly hardened piece of tool steel may be magnetized nearly normally, but usually irregular hardening leads to an irregular distribution of the magnetism which shows itself in an abnormal iron-filing diagram. An unusual filing diagram does not, however, as some instances given show, always indicate that the distribution of the magnetic induction in the bar is very irregular.

My thanks are due to the Trustees of the Bache Fund of the National Academy of Sciences who have kindly lent me some of the apparatus used in making the observations described in this paper.

XVIII

THE MAGNETIC PERMEABILITIES AT LOW EXCITATIONS OF TWO KINDS OF VERY PURE SOFT IRON [1]

More than a year ago, I had occasion to study the magnetic properties under very high excitations of a piece of Norway iron [2] (P), which proved when analyzed to be extraordinarily pure. The tests made in the Chemical Laboratory of Harvard University by Mr. E. R. Riegel for nickel, cobalt, tungsten, and even manganese, as well as for the metals of Groups IV and V, were all negative. There was less than 0.03 per cent of carbon, less than 0.047 per cent of phosphorus, less than 0.03 per cent of silicon, and less than 0.003 per cent of sulphur. A slender rod of this remarkable iron, of which we had originally a round bar five centimeters in diameter and thirty-four centimeters long, had, when annealed, an extremely high permeability under excitations above 200, but, because of the local reluctance at the joints, it did not prove easy to determine the permeability of this rod in a yoke at low excitations. The metal showed to the eye a fibrous structure with striae parallel to the length of the bar, as if minute quantities of scale had been included in the bar in the rolling; and it seemed likely that the specific reluctance to magnetization across the grain of the iron would be greater than to magnetization parallel to the grain. Under these circumstances it was probable that the permeability of a ring, so cut from the metal that its axis should be parallel to the grain, would appear low. It happened, however, that I had two such rings, but that there was not enough of the iron left to make rings with axes perpendicular to the grain, and I was forced to get what information I could from them, though it soon became evident that for excitations above five gausses the permeability fell below what commercial Norway iron should show.

This paper gives the results of some tests made at low excitations, which are interesting because of the great susceptibility which the

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlvi, no. 9, Oct. 1910.

[2] *American Journal of Science*, vol. xxviii, July, 1909.

rings showed in fields less than two gausses, and compares the magnetic behavior of this metal with that of a ring of the so-called "American Ingot Iron," which I obtained through the kindness of Dr. P. W. Bridgman. This well-known iron, which was made by the American Rolling Mill Company of Middletown, Ohio, seems to be perfectly homogeneous, and, according to the makers, contains less than 0.03 per cent of impurities all told.

All the rings were very accurately made by Mr. G. W. Thompson, the mechanician of the Jefferson Laboratory. The external diameters of the Norway iron rings were 5.000 cm. and 4.996 cm. respectively; their thicknesses were 0.250 cm. and 0.254 cm., and their breadths were 1.2204 cm. and 1.210 cm. The measurements were made with the help of Zeiss Comparator No. 3196 and a set of auxiliary gauges. After each ring had been measured, a coil of very fine double-silk-covered copper wire was wound on the metal in a single layer and then baked in shellac. Over this was wound, usually in two layers, the exciting coil of well-insulated wire nearly one millimeter in diameter. The ballistic galvanometers were of the moving coil type, and had periods amply long enough [1] for the work. The fine coil on the ring was always in simple circuit with the galvanometer and the secondary coil of a standard of self inductance tested by the Bureau of Standards.

The maximum value of the permeability (5480) which I obtained for the first ring tested seemed so high that at first I suspected that there was some error in the determination, so I changed the galvanometer, and then took off the coils and wound on new ones with different numbers of turns; but when the result was unchanged and the second ring gave values for the ordinates of the HB diagram which were practically indistinguishable from those obtained from the first ring, there seemed to be no doubt that the work had been accurately done. The two rings lay side by side in the original bar, and both must have had nearly the same discontinuities. Table I, founded upon several hundred separate determinations, gives values of the permeability of the metal obtained from 35 different excitations of the first ring and 25 of the second. A ring of very pure annealed iron from the Armstrong Works at Elswick gave in the hands of Wilson the

[1] Peirce, *Proceedings of the American Academy*, vol. xliv, 1909. (283).

same maximum value of the permeability as the rings just mentioned; but apart from the reports of some tests upon thin pieces of electrolytically deposited iron, I have found no other records of permeabilities so high as this.[1] For excitations above six gausses, however, the rings

TABLE I

ANNEALED RING OF NORWAY IRON (P), AXIS PARALLEL TO THE AXIS OF THE ORIGINAL BAR. MEASUREMENTS MADE BY THE METHOD OF ASCENDING REVERSALS

H	*B*	*B/H*	*I*	*H*	*B*	*B/H*	*I*
0.05	22	440	1.7	1.70	9040	5320	719
0.10	54	540	4.3	1.80	9340	5190	743
0.15	92	613	7.3	1.90	9640	5070	767
0.20	145	725	11.3	2.00	9870	4935	785
0.25	205	820	16	2.20	10300	4680	820
0.30	280	933	22	2.40	10690	4450	851
0.35	390	1114	31	2.60	11010	4230	876
0.40	495	1238	39	2.80	11310	4040	900
0.45	620	1377	49	3.00	11560	3850	920
0.50	790	1580	63	3.50	12010	3430	956
0.55	990	1800	79	4.00	12310	3080	980
0.60	1260	2100	100	4.50	12550	2790	991
0.65	1585	2440	126	5.00	12760	2550	1016
0.70	1980	2830	158	6.00	13060	2180	1039
0.75	2405	3205	191	7.00	13300	1900	1058
0.80	2850	3560	227	8.00	13510	1690	1075
0.85	3310	3890	263	9.00	13700	1520	1090
0.90	3780	4200	301	10.00	13870	1387	1103
0.95	4255	4480	339	20.00	14950	748	1188
1.00	4730	4730	376	30.00	15520	517	1233
1.10	5620	5110	499	40.00	15850	396	1258
1.20	6380	5320	508	50.00	16080	322	1276
1.30	7070	5440	563	60.00	16300	271	1292
1.40	7670	5480	610	70.00	16510	236	1308
1.50	8200	5470	653	80.00	16700	209	1322
1.60	8640	5400	687				

were distinctly less permeable than good iron should be, and this anomalous behavior is perhaps due to discontinuities across the lines of magnetization.

[1] Stoletow, *Ann. d. Physik*, vol. cxlvi, 1872 (439); Riecke, *Ann. d. Physik*, vol. cxlix, 1873; Rowland, *Phil. Mag.*, xlvi, 1873 (140); Roessler, *Inaugural Diss.*, Zürich, 1892; Holz, *Ann. d. Physik*, vol. viii, 1876 (353); Ewing, *Magnetic Induction in Iron and Other Metals*. Bauer, *Ann. d. Physik*, vol. xi, 1880 (349); G. vom Hofe, *Ann. d. Physik*, vol. xxxvii, 1889 (482); Lehmann, *Ann. d. Physik*, vol. xlviii, 1893 (406); Benedicks, *Ann. d. Physik*, vol. vi, 1901; Lydall and

A well-annealed isthmus of this iron cut lengthwise of the bar gave for I, under excitations as high as 18000 gausses, a final value of 1795, and an unannealed rod tested in a yoke gave 1730. These remarkable values point to a much higher permeability at medium excitations than the rings just mentioned show.

Table II shows corresponding values of H and B for a ring of annealed "American Ingot Iron" cut out by Mr. Thompson from a large

TABLE II

Ring of Annealed "American Ingot Iron." Measurements made by the Method of Ascending Reversals

H	B	B/H	I	H	B	B/H	I
0.1	35	175	2.8	4.0	11900	2975	947
0.3	106	530	8.4	4.5	12450	2777	991
0.2	198	660	15.7	5.0	12850	2570	1022
0.4	324	810	26	5.5	13200	2400	1050
0.5	482	964	38	6.0	13500	2250	1074
0.6	688	1147	55	6.5	13780	2120	1096
0.7	990	1414	79	7.0	13960	1994	1111
0.8	1325	1656	105	7.5	14180	1891	1128
0.9	1720	1911	137	8.0	14350	1794	1142
1.0	2320	2320	185	8.5	14480	1763	1152
1.5	5150	3433	410	9.0	14600	1622	1162
2.0	7340	3670	584	9.5	14720	1550	1171
2.5	9050	3620	720	10.0	14830	1483	1180
3.0	10220	3406	813	20.0	15850	792	1261
3.5	11150	3186	887				

plate of the metal. The outside diameter of the ring was 5.012 cm., its thickness was 0.283 cm., and its breadth about 2.116 cm. There were 112 turns in the testing coil and 74 turns in the exciting coil.

Table III gives for comparison the results of the determinations of the permeabilities of a number of different specimens of soft iron by different observers. Some of the numbers which I have obtained graphically from the published figures are only approximately correct.

Columns 7 and 9 give the records of observations made upon two small rings of very pure iron given by Colonel Dyer of the Elswick works to Sir Frederick Abel, who presented them to Dr. John Hop-

Pocklington, *Proc. Roy. Soc.*, vol. lii, 1892; Wilson, *Proc. Roy. Soc.*, vol. lxii, 1898; du Bois, *Ann. d. Physik*, li, 1894 (537); Taylor Jones *Ann. d. Physik*, liv, 1895 (641); *Phil. Mag.*, xxxix, 1895 (254); Stefan, *Wiener Berichte*, lxxxi, 1880 (89); Ewing, *Roy. Soc. Trans.*

kinson. The tests upon the first ring by Messrs. Pocklington and Lydall seem to show that they did not anneal the iron; the remarkable measurements of Wilson upon the second ring were made after the iron had been softened. Norway Iron (R) was a long annealed rod about half an inch in diameter. This was tested in a solenoid.

TABLE III

H	B											
	Norway Iron (P) Rings, 1 and 2	Long Rod of Norway Iron (R.)	American Ingot Iron	Hopkinson's Annealed Wrought Iron (Average)	Lehmann's Swedish Iron Ring	Rössler's Iron Ellipsoid	Wilson's Ring of Pure Iron (No. 2)	Ewing's Soft Iron Wire	Lydall and Pocklington's Ring of Pure Iron (No. 1)	Hopkinson's Ring	Rowland's Norway Iron Ring	Norway Iron (Q)
0.2	145	90	106	110	140	68	300	26	46	80	100	115
0.4	495	260	324	340	450	165	800	55	128	170	380	340
0.6	1260	570	688	740	1180	282	1400	100	467	330	1100	780
0.8	2850	1125	1325	1280	2050	410	2050	160	1020	550	2500	1750
1.0	4730	2150	2320	2400	3020	560	3400	230	2150	950	4500	3050
1.5	8200	4600	5150	4700	4700	1550	8000	500	6020	3650	6950	5450
2.0	9870	6600	7340	5900	6050	4350	10700	900	7720	5400	8650	7200
2.5	10850	8240	9050	6800	7200	6950	12000	2500	8920	6780	9900	8200
3.0	11560	9480	10220	7600	8300	8700	12800	5800	9850	7800	10800	9150
3.5	12010	10460	11150	8350	9150	9800	13400	8400	10600	8800	11450	10050
4.0	12310	11280	11900	9000	9700	10600	13800	10350	11300	9600	12020	10800
4.5	12550	11980	12450	9500	10150	11240	14150	11650	11850	10250	12500	11400
5.0	12760	12560	12850	9950	10650	11800	14360	12480	12350	10750	12070	11900
6.0	13060	13400	13500	10700	11400	13700	14700	13500	13200	11500	13700	12800
7.0	13300	13900	13960	11300	12050	13400	14950	14100	13900	12100	14100	13400
8.0	13510	14300	14350	11800	12550	13780	15100	14350	14400	12550	14400	13850
9.0	13700	14660	14600	12180	12900	13960	15250	14750	14800	12950	14680	14200
10.0	13870	14940	14830	12500	13280	14050	15320	14950	15050	13350	14950	14400

An analysis of Hopkinson's ring made by the Whitworths, showed manganese 0.143 per cent, phosphorus 0.271 per cent, sulphur 0.012 per cent, carbon 0.01 per cent and "slag" 0.436 per cent. The Elswick iron contained 0.1 per cent of manganese and 0.013 per cent of sulphur but no phosphorus and hardly a trace of carbon or other impurity. Norway Iron (Q) was an annealed ring cut from a bar of "pure iron" obtained in the Boston market.

My thanks are due to Professor John Trowbridge, who furnished me with the pure iron described above, and to the Trustees of the Bache Fund of the National Academy of Sciences who loaned me some of the apparatus used in the work.

XIX

THE EFFECTS OF SUDDEN CHANGES IN THE INDUCTANCES OF ELECTRIC CIRCUITS AS ILLUSTRATIVE OF THE ABSENCE OF MAGNETIC LAG AND OF THE VON WALTENHOFEN PHENOMENON IN FINELY DIVIDED CORES. CERTAIN MECHANICAL ANALOGIES OF THE ELECTRICAL PROBLEMS [1]

In making some kinds of electrical measurements, one occasionally needs to alter abruptly the inductances of a circuit and to inquire what the effect of the change is upon the march of the currents which the circuit is carrying. If the circuit happens to be a simple one with no magnetic metals and no other circuits near, and if the whole change takes place in a sufficiently short time, it is easy to compute the magnitude and the direction of the corresponding change in the current. If, however, the circuit is complex, or affected by the presence of other inductive circuits in the neighborhood, and if the duration of the change in inductance is long compared with the various time constants which enter, the problem may be much more difficult; though if there be no magnetic metals in the field, the principles laid down more than forty years ago by Maxwell [2] in his dynamical theory of the electromagnetic field, and soon afterwards elaborated and illustrated by Rayleigh and others, point the way to the solution.

In most cases which present themselves in practice, there are masses of magnetizable metal in the form of cores, near the circuits to be studied, and it is often difficult, even if one knows something about the magnetic properties and the history of the cores, to predict exactly what the effects of a given sudden change in the inductances will be. This paper discusses first, with the help of mechanical analogies, a few simple and familiar cases of circuits without cores, with the purpose of emphasizing some facts to be met with also when cores are present, and then gives a number of diagrams obtained from the photographic records of oscillographs in circuits which contained

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlvi, no. 20, April, 1911.

[2] Maxwell, *Phil. Trans.*, Dec. 1864; Rayleigh, *Phil. Mag.*, vol. xxxviii, 1869, vol. xxxix, 1870, vol. xxx, 1890.

large electromagnets some with solid and some with divided cores. Changes in the inductances of a circuit which contains one or more electromagnets often involve the moving of comparatively large masses of metal, and it is obviously impossible to make such changes instantaneously even though they may be carried out in intervals which are not long relatively to the time constants of the circuit. In solid cores, also, eddy currents tend to mask the effects of sudden changes in the conformation of the circuit, and this, with the fact that the susceptibility of the iron depends not only upon the intensity of the present excitation, but also upon the past experiences of the metal, leads to considerable differences in the magnetic behavior of a circuit according as it does or does not "contain iron." The diagrams show these differences and illustrate some typical conditions which arise in practical work.

It is well known that the final flux of magnetic induction through a solid iron core is not determined by the intensity of the excitation alone even when its magnetic condition at the outset is given, but depends in many cases upon the manner of application of the given excitation — whether it be made suddenly, by small steps at intervals, or by slow, continuous rise. The diagrams are interesting in this connection because they show that when the core is fairly well divided, the forms of two distinct portions of a current curve, interrupted by a sudden change of inductances, are often almost identical with corresponding portions of two current curves obtained without any such interruption, the one with the original inductances, the other with the final ones.

It may be well to consider briefly at the outset the very simplest case, the familiar one of a single circuit, without iron, of fixed resistance, r ohms, and of inductance originally equal to L_0 henries, which contains a constant electromotive force of E volts and is carrying at the time $t = 0$, a current of C_0 amperes. At this instant ($t = 0$) let the inductance begin to change according to some law, and progressing always in the same direction, let it attain at the time T the given value L_1, after which it shall remain constant.

In order to illustrate graphically the effect of making the given change in inductance in longer or shorter time intervals, it will be

convenient to use three rectangular axes for t, C, and T respectively, and to represent the value (L) of the inductance at any time $0<t<T$, by the expression

$$L_0 + (L_1 - L_0) f\left(\frac{t}{T}\right), \tag{1}$$

where $f(0) = 0, f(1) = 1$, and for $0<x<1, f'(x) > 0$.

The line OK in the tT plane (Figure 1) has the equation $t = T$, and after the value ($T = OF$) has been fixed for T, the course of the current during the change of inductance may be shown by a curve

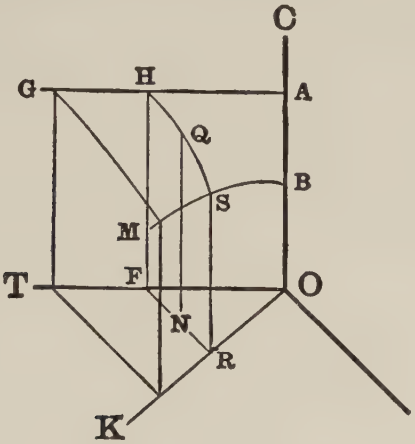


FIGURE 1

(HQS) in a plane (FRS), the equation of which is $T = T'$. FH, which measures the ordinates of the line AG in the CT plane represents the initial current (C_0) at the time $t = 0$, and the ordinates of the curve MB in the plane COK represent for different values of T the intensities of the current at the end of the change in inductance, when the form of the function f is the same. The vertical distance AB shows the magnitude of the sudden change in the current strength when the change in L is supposed to be instantaneous and is the same whatever form f has.

During the interval $0<t<T$, the current is to be found, of course, by solving the equation

$$E - \frac{d(LC)}{dt} = rC, \tag{2}$$

— where L has the variable value given above — and making the constant of integration such that $C = C_0$ when $t = 0$. After the epoch $t = T$, the intensity of the current satisfies the equation

$$E - L_1 \cdot \frac{dC}{dt} = rC, \tag{3}$$

in which the coefficients are constants. Equation (2) may be written in the form

$$\frac{dC}{dt} + \frac{r + L'}{L} \cdot C = \frac{E}{L}, \tag{4}$$

where $L' = dL/dt$, and if we make use of the usual notation,[1] and put $P = (r + L')/L$, $Q = E/L$, we shall have in general

$$C = e^{-p}\left(M + \int_0^t Qe^r dt\right), \tag{5}$$

where

$$p = \int_0^t P \cdot dt = \log\left(\frac{L}{L_0}\right) + \int_0^t \frac{r dt}{L} \cdot \tag{6}$$

That is, if $0 \leqq t \leqq T$,

$$C_t = \frac{L_0}{L_t} e^{-\int_0^t \frac{r dt}{L}} \left(C_0 + \frac{E}{L_0} \int_0^t e^{\int_0^t \frac{r dt}{L}} \cdot dt\right), \tag{7}$$

and, in particular,

$$C_T = \frac{L_0}{L_1} e^{-\int_0^T \frac{r dt}{L}} \left(C_0 + \frac{E}{L_0} \int_0^T e^{\int_0^t \frac{r dt}{L}} \cdot dt\right). \tag{8}$$

In this expression L is to progress always in the same direction from L_0 to L_1, and cannot pass through the value zero, so that the limit of C_T as T approaches zero has the familiar value

$$\underset{T = 0}{\text{Limit}}\ C_T = \frac{L_0 C_0}{L_1}, \tag{9}$$

which might have been found directly by integrating (2) with respect to t from 0 to T; the electromagnetic momentum has no sudden change. Equation (9) follows immediately, of course, when one makes use of the usual analogies between the phenomena of ordinary me-

[1] Forsyth, *Treatise on Differential Equations*, § 14.

chanics and those of electromagnetism. Equation (2) is in form like the equation of motion of a system the mass of which changes with the time in a certain given manner and which is under the action of a constant accelerating force and a retarding force proportional to the velocity. Let a moving mass L grow steadily during its motion by the gradual accretion of small particles which, originally at rest, are suddenly made part of the moving system, much as the links of a fine chain which has been lying on a table are successively set in motion when one end of the chain is lifted more and more; or let the mass L decrease steadily by the loss of small particles each of which leaves the system with a parting push which reduces its own velocity to zero and speeds its late companions on their way; then, if C is the velocity

FIGURE 2

of the moving mass, E the accelerating force, and rC the retarding force, the equation of motion will be $d(LC)/dt = E - rC$, that is, (2).

If, when its velocity is C_0, the mass of such a system be instantaneously changed from L_0 to L_1, the principle of the conservation of momentum in impact shows that if C_1 is the velocity immediately after the impulsive change, $L_0C_0 = L_1C_1$.

The conventional diagram shown in Figure 2 indicates the nature of this simple mechanical problem. L_0 is a mass furnished with a stiff vane of such a size as to make the air resistance (which is proportional to the velocity) equal to r units when the mass is moving with unit velocity. L_0 is urged to the right by the constant force E and is retarded by a force rv. A slack inextensible string connects L_0 with another mass $L_1 - L_0$, and when the string becomes taut, the impulsive change in the velocity of L_0 corresponds to the change in the current in the inductive circuit when the inductance is impulsively changed from L_0 to L_1.

If the induction flux, N, in a circuit which contains no iron be plotted against the current, the resulting locus is a straight line

through the origin, the slope of which is the self-inductance of the circuit. If, then, the lines (OH, OV, Figure 3) corresponding to L_0 and L_1 be drawn, and if when the rising current has attained the value C_0, the inductance be supposed to change suddenly to L_1, the induction flux through the circuit preserves its value unchanged while the current falls from C_0 to C_1, and the point in the diagram which gives the state of the circuit moves from F to T.

If, as is approximately the case with some circuits which have open cores made of very finely divided soft iron, the hysteresis diagram is

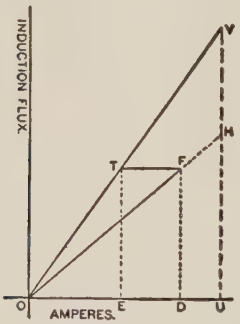


FIGURE 3. — The line OFTV represents the change of the induction flux linked with a circuit without iron, when the inductance is suddenly increased.

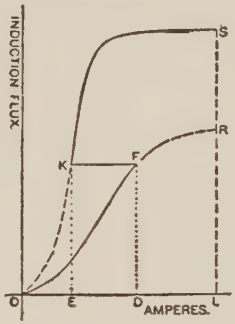


FIGURE 4. — The line OFKS represents the growth of the induction flux in a magnet with finely divided core when the inductance of the circuit is suddenly increased.

extremely narrow, so that the inductance may be considered to be a definite function of the current strength, we may represent two different states of the circuit by lines like OFR and OKS of Figure 4. It is then easy to see that in this case also a sudden change from one state to the other when the current had the value OD = C_0 would leave the induction flux through the circuit momentarily unchanged while the current fell to OE = C_1, and the point which represents the state of the circuit would suddenly move from F to K.

If, at any instant, the total flux of magnetic induction through any simple circuit, which may or may not contain iron, is N (maxwells), if r is the resistance of the circuit in ohms, C, the current in amperes, and E, the applied electromotive force in volts.

$$E - \frac{1}{10^8} \cdot \frac{dN}{dt} = rC, \tag{10}$$

or

$$\frac{dN}{dt} = 10^8 \cdot r\left(\frac{E}{r} - C\right), \tag{11}$$

and if the final value (E/r) of the current be denoted by C', and the change in N during the time interval t_1 to t_2 by $N_{1,\,2}$,

$$N_{1,\,2} = r \cdot 10^8 \int_{t_1}^{t_2} (C' - C)\, dt. \tag{12}$$

If, now, C be plotted against the time (as in an oscillograph diagram) in a curve s (Figure 5) [1] in which l centimeters parallel to the axis of abscissas represent one second, and an ordinate m centimeters long one ampere, the curve will have a horizontal asymptote (CY) at a distance (KC) corresponding to E/r amperes from the time axis, and, if OK represents the time t_1 and OL the time t_2, the area FGDC, or $A_{1,\,2}$, expressed in square centimeters, is equal to

$$lm \int_{t_1}^{t_2} (C' - C)\, dt, \tag{13}$$

and

$$N_{1,\,2} = \frac{10^8}{lm}\,(r \cdot A_{1,\,2}). \tag{14}$$

The curve ONJ of Figure 6, which has been carefully drawn to scale, represents the growth of the current with the time in a circuit without iron, of resistance r and inductance L. The curve OPT represents the current in the same circuit when the inductance has been increased to $4L$, while the resistance is the same as before. If, when the inductance of the circuit is L, the current rises in the time OU to the value UN, and if then the inductance is instantly increased to $4L$, the current falls to UF and then rises again in the manner indicated by the curve FG, which is the curve OPT moved to the right through a distance OL just great enough to make its ordinate at the time OU equal to one fourth of UN. Since the area between the curve and its asymptote is proportional to the inductance flux through the circuit, it is clear without any of the reasoning of the preceding paragraphs, that there cannot be any impulsive change of the induction flux when

[1] Identical with Fig. 3 on p. 106.

the inductance is suddenly increased. A glance at the figure shows, however, that the rate of increase of the induction suddenly becomes much greater than it was just before the change.

Curve ODE of Figure 7 shows the manner of growth of the current in another simple circuit of fixed inductance, $4L$. If, at the time OW, when the current has attained half its final strength, and the induction

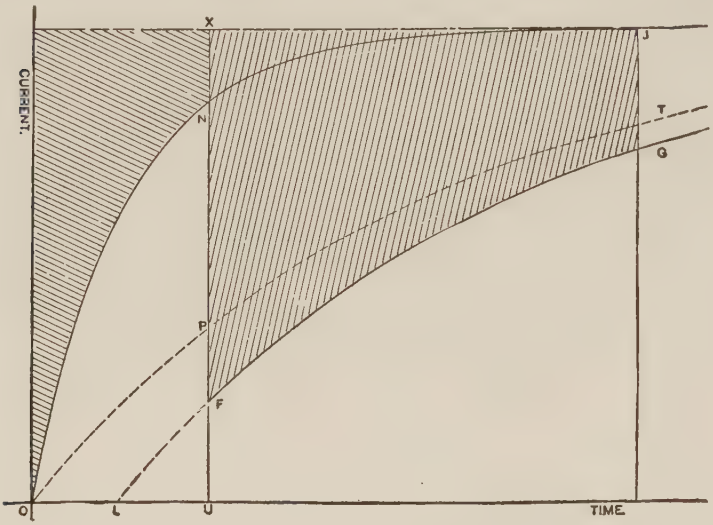


FIGURE 6. — The line ONJ represents the current in a circuit of inductance L without iron. OPT shows the form of the current in the same circuit when the inductance has been increased to $4L$. ONFG is the current when the inductance is suddenly changed from L to $4L$ at the time OU.

flux through the circuit is represented on the scale indicated by equation OABD, the inductance be suddenly changed to L, the current suddenly becomes four times as strong as it was and then falls in a manner shown by the curve ST. The flux through the circuit just after the change is already twice as large as it will be eventually when the current reaches its final value, OA, and it decreases by an amount represented by the area BST, which is half the area AODB. Just before the change the flux was increasing with the time at a rate represented by the length of the line BD; just after the change it decreases at a rate represented by the line BS, which is twice as long as DB.

In the case of a circuit which does not contain iron, an increase of inductance without an increase of the resistance usually involves a change of the conformation of the circuit, and this generally requires a considerable fraction of a second, at least, to bring about, so that the formula (9) cannot be used to determine the current strength at the end of the inductance change. To illustrate this fact we may assume

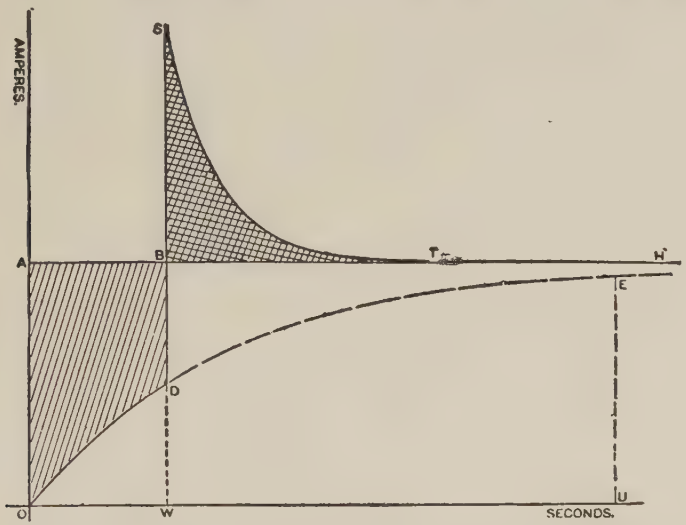


FIGURE 7. — ODE shows the current in a simple circuit of fixed inductance, $4L$. If at the time OW, when the current has attained half its final intensity, the inductance is suddenly reduced to L, the course of the current will be ODBST.

that the change from L_0 to L_1 in the time T is brought about at a constant rate so that $L = L_0 + t(L_1 - L_0)/T$, and the strength of the current at the time t is given by the equation

$$C = \left(\frac{L_0}{L}\right)^m \cdot \left(C_0 + \frac{ET}{m\,(L_1 - L_0)} \cdot \frac{L^m - {L_0}^m}{{L_0}^m}\right), \tag{15}$$

where

$$m = \frac{rT + L_1 - L_0}{L_1 - L_0}. \tag{16}$$

If, now, $C = E/r$ amperes, $L_0 = 2$ henries, $L_1 = 4$ henries, then, according as T is one second, half a second, one tenth of a second, or one hundredth of a second, the value (C_1) of the current at the end of the interval T is $0.980 \cdot C_0$, $0.962 \cdot C_0$, $0.836 \cdot C_0$, or $0.569 \cdot C_0$, whereas C_1

would be $0.5 \cdot C_0$, if the change in the inductance had been instantaneous.

Figure 8 shows in TW the relative changes in the current in this circuit from $t = 0$ to $t = T$, when T is one tenth of a second, and in TZ the changes when T is one one-hundredth of a second. If the

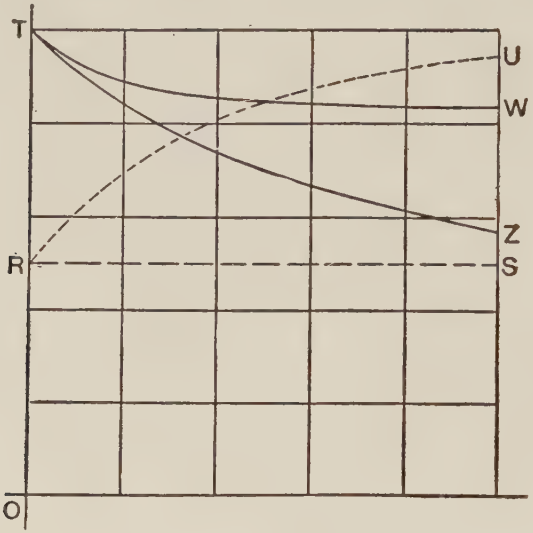


FIGURE 8. — If in a certain inductive circuit, without iron, the inductance be instantaneously doubled, the course of the current in the next tenth of a second will be TRU. TW shows the current if the doubling be brought about by a continuous change going on at a constant rate during the whole interval. TZ shows on a different time scale, the course of the current for a hundredth of a second, if during this interval the inductance be changed at a constant rate which results at the end in its being doubled.

change were instantaneous the course of the current in one tenth of a second would correspond to the line TRU.

We may next consider the somewhat less simple circuit indicated in Figure 9_a, consisting of three parallel branches each of which has self-inductance, but no two of which have mutual imductance. Let r, r_1, r_2 be the resistances of the branches, L, L_1, L_2 their inductances, E, E_1, E_2 the constant electromotive forces of the generators in them, and C, C_1, C_2 the currents. At the time $t = 0$, when the currents and the induc-

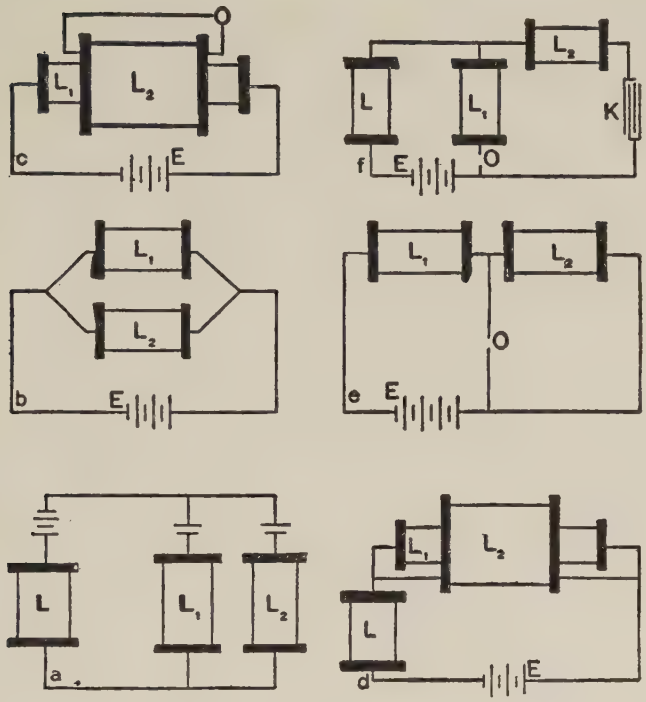


FIGURE 9

tances have given values, let the inductances begin to change according to given laws each of which can be expressed by an equation similar to (1), and let them attain, at the time T, other given values, which they thereafter keep. It is evident that any instant during the interval $0 < t < T$

$$E + E_1 - \frac{d(LC)}{dt} - \frac{d(L_1C_1)}{dt} = rC + r_1C_1,$$

$$E + E_2 - \frac{d(LC)}{dt} - \frac{d(L_2C_2)}{dt} = rC + r_2C_2, \tag{17}$$

$$C = C_1 + C_2,$$

or, if we represent differentiation with respect to the time by accents,

$$(L + L_1)C'_1 + LC'_2 + (L' + L'_1 + r + r_1)C_1 + (L' + r)C_2 = E + E_1,$$

$$LC'_1 + (L + L_2)C'_2 + (L' + r)C_1 + (L' + L'_2 + r + r_2)C_2 = E + E_2. \tag{18}$$

If, from these equations and others obtained by differentiating them with respect to the time, C_2 and its derivatives be eliminated, we shall get a differential equation of the second order for C_1 in which the inductances and their derivatives are known functions of t, and the initial values of C_1 and C'_1 are also known. This new equation may be found by equating to zero the determinant

$$\begin{vmatrix} L & 2L'+r & L'' & (L+L_1)C''_1+(2L'+2L'_1+r+r_1)C'_1+(L''+L''_1C_1 \\ L+L_2 & 2L'+2L'_2+r+r_2 & L''+L''_2 & LC''_1+(2L'+r)C'_1+L''C_1 \\ 0 & L & L'+r & (L+L_1)C''_1+(L'+L'_1+r+r_1)C_1-E-E_1 \\ 0 & L+L_2 & L'+L'_2+r+r_2 & LC'_1+(L'+r)C_1-E-E_2 \end{vmatrix} \quad (19)$$

and, although it may be somewhat simplified, it generally proves rather intractable. If, however, the interval T is so short that the changes in the inductances may be regarded as impulsive, the corresponding changes in the currents may be found immediately, for if the equations be integrated with respect to the time from $t = 0$ to $t = T$, and if T be made to approach zero, while the currents remain finite, it appears that $LC + L_1C_1$ and $LC + L_2C_2$ have the same values just after the impulsive change in the inductances as they had just before the change. The induction flux through each circuit chosen for the equations remains unchanged by the sudden change of inductances.

It is easy to find a number of different problems in mechanics each of which yields equations of motion of the form (17), and is, therefore, analogous in a sense to the electromagnetic problem under consideration. Such an analogy, even though it be difficult to embody it in a working model, sometimes makes clearer to a person already familiar with mechanical principles the nature of the phenomena which he is to look for in interpreting his electrical equations. It will do no harm if, in imagining a mechanical system which is to serve this purpose, we postulate the existence of flexible, inextensible, massless strings, or even, at a pinch, the existence of stiff, nearly massless rods, or of pulley wheels so light that their moments of inertia shall be negligible. It is often desirable to imagine the motions of the masses which in the mechanical system represent the inductances in the electrical problem, to be hindered by retarding forces proportional to the velocities, to represent the electrical resistances. The resistance which the air offers to a body moving through it with a constant velocity not greater than 50

cms. per second is very nearly proportional to that velocity; and since the velocities which in the mechanical case correspond to the currents are usually much smaller than that, the resistance may be sufficiently well indicated by thin wings or vanes of proper size attached to the masses.

In the arrangement shown in Figure 10 the masses L, L_1, L_2, are urged towards the bottom of the diagram by forces of intensity E, E_1,

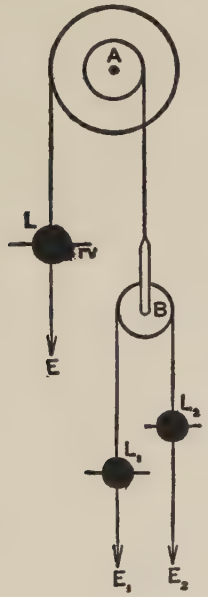


FIGURE 10

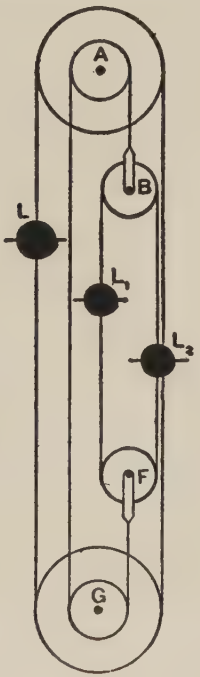


FIGURE 11

E_2. The lines drawn across the masses indicate wings of such shapes as to make the resistances due to the air r, r_1, r_2 dynes respectively, when the corresponding velocities are one centimeter per second. It is evident from the geometry of the figure that the velocity of L downward is equal to the sum of the velocities of L_1 and L_2 upward. The tension of the string attached to L and passing over the massless pulley A is at every instant half that of the cord which is attached to the massless pulley B, and equal to the tension of the cord which con-

nects L_1 and L_2. The equations of motion of the masses are of the form (17). If, as a consequence of applied forces or impulses, the string should become slack, the analogy between the mechanical and the electromagnetic problems would disappear, and it is sometimes convenient to imagine the masses attached to taut endless strings in

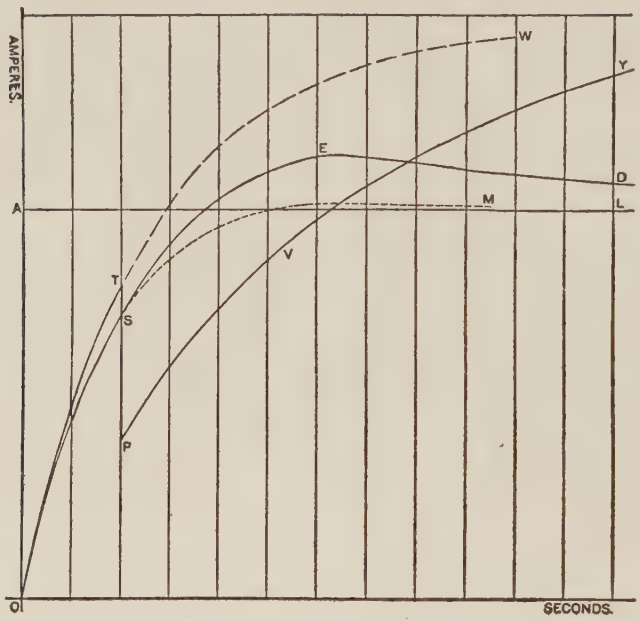


FIGURE 12. — The lines OTW, OSM, show the forms of the currents in two parallel inductive resistances which connect the terminals of a storage battery. When at a given instant, the inductance of one of the parallel branches is suddenly doubled, the current in it changes its value abruptly and takes the course OTPVY, while the current in the other (OSED) suffers no sudden change in strength.

some such manner as is shown in Figure 11. It is very easy to construct a model of this kind which will work fairly well if one uses for masses properly loaded roller skates which move about on the level top of a table. The masses may be connected by fine catgut passing around small, cheap pulleys with vertical axes mounted on the table.

A special case of some practical interest is that indicated in Figure 9_b, where the terminals of a battery without sensible self-inductance

are connected by two inductive branches in parallel. The currents are given by the equations

$$\begin{aligned} L_1L_2\frac{d^2C_1}{dt^2}+[(r+r_1)L_2+(r+r_2)L_1]\frac{dC_1}{dt} \\ +(r_1r_2+rr_1+rr_2)C_1=r_2E \\ L_1L_2\frac{d^2C_2}{dt^2}+[(r+r_1)L_2+(r+r_2)L_1]\frac{dC_2}{dt} \\ +(r_1r_2+rr_1+rr_2)C_2=r_1E, \end{aligned} \tag{20}$$

and it is clear that if the inductances are suddenly changed, the products L_1C_1 and L_2C_2 are continuous, and if, in particular, only one of the inductances is altered, the current in the parallel branch is itself continuous. Figures 12 and 13 are drawn to scale for two typical cases which indicate well enough what is usually to be expected. In both diagrams $L_1 = 1$, $L_2 = 1$, $r = 12$, $r_1 = 20$, $r_2 = 30$, $E = 120$ (or these quantities are to be in the proportions here given). The final values of C_1 and C_2 are 3 amperes and 2 amperes.

In the case which corresponds to Figure 12 the battery circuit is closed at a given instant, and 0.02 seconds afterwards, when C_1 has attained the value 1.607 and C_2 the value 1.457, L_1 is suddenly changed from 1 to 2. As a consequence, C_1 falls suddenly to 0.8035, while C_2 remains momentarily unchanged. Before the change, the currents were given by the equations

$$C_1 = 3 - \tfrac{120}{104}e^{-24t} - \tfrac{120}{65}e^{-50t}, \quad C_2 = 2 + \tfrac{120}{156}e^{-24t} - \tfrac{360}{130}e^{-50t}, \tag{21}$$

and afterwards by the approximate equations

$$\begin{aligned} C_1 &= 3 - 1.912\,e^{-13.48t} - 0.284\,e^{-44.52t}, \\ C_2 &= 2 + 0.803\,e^{-13.48t} - 1.346\,e^{-44.52t}. \end{aligned} \tag{22}$$

The line OTPVY shows the course of C_1, and OSED the course of C_2. It will be observed that C_2 approaches its final value from above.

If the change in inductance is made after the currents have attained their final values, the courses of C_1 and C_2 will be those indicated in Figure 13 by the lines KGQS and LRDT. If after the currents have reached their steady values, the main circuit be suddenly broken, C_1

and C_2 instantly acquire equal and opposite values, and the subsequent course of C_1 is given by the equation

$$C_1 = \frac{E\,(L_1r_2 - L_2r_1)}{r_1r_2 + rr_1 + rr_2}\,e^{-\frac{r_1+r_2}{L_1+L_2}t}. \tag{23}$$

See in Figure 13 the line GRXZ.

If the terminals of an open battery circuit of inductance L and of resistance r be connected by a number of inductive conductors in parallel, of resistances r_1, r_2, r_3, r_4, etc., and of inductances L_1, L_2, L_3, L_4, etc., and if sudden changes be made in the inductances, the quantities

$$LC + L_1C_1,\ LC + L_2C_2,\ LC + L_3C_3,$$

etc., will be continuous. If L is negligible, and if only some of the other inductances be impulsively changed, the currents in the other branches will be continuous.

If, in the arrangement shown in Figure 14, the masses P, Q, R are numerically equal to L_1, L, L_2, respectively, if the velocities of P and R in the direction of the bottom of the page are C_1, C_2, and if the dimensions of the vanes attached to the masses are such that the air offers resistance of r_1, r, r_2 times the velocities to the motion of P, Q, and R, the equations of motion of the masses are identical with the current equations for the electrical circuit shown in the figure.

The currents in two neighboring circuits (Figure 9$_c$) of self-inductances L_1, L_2, and mutual inductance M, which contain the electromotives forces E_1, E_2, are given by the familiar equations

$$\begin{aligned} E_1 - L_1\frac{dC_1}{dt} - M\frac{dC_2}{dt} - r_1C_1 &= 0, \\ E_2 - L_2\frac{dC_2}{dt} - M\frac{dC_1}{dt} - r_2C_2 &= 0, \end{aligned} \tag{24}$$

and any impulsive changes in the inductances cause such sudden changes in the current as will keep $L_1C_1 + MC_2$ and $L_2C_2 + MC_1$ momentarily unchanged.

Many different working models have been made to illustrate the simple electrical problems which concern two such circuits. Of these

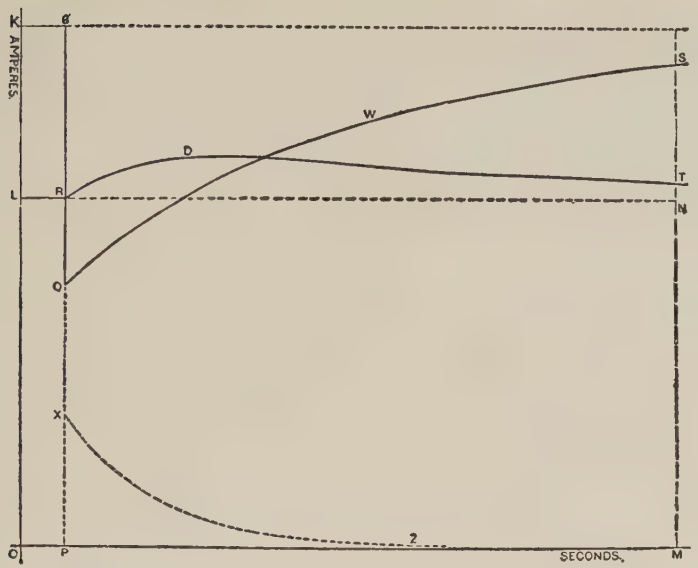


FIGURE 13. After the currents in two parallel inductive resistances which connect the terminals of a storage battery have become steady, at the values OK, OL, the inductance of one of the branches is suddenly doubled so that the current in it takes the course KGQWS. The current in the other branch takes the continuous form LRDT and approaches its final value from above.

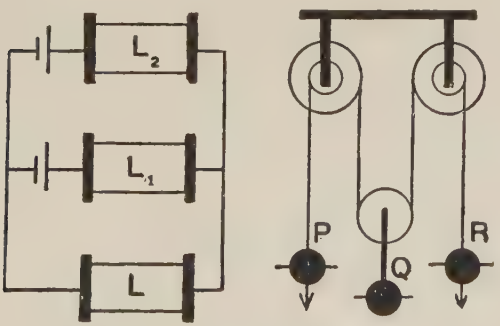


FIGURE 14

some of the best known are due to Maxwell, Rayleigh, J. J. Thomson, Webster, and Boltzmann.

The original model of Maxwell, now in the Cavendish Laboratory, is represented by Figure 15$_a$, taken from Gray's Absolute Measurements in Electricity and Magnetism, where an excellent account of the apparatus and its theory may be found.

In Lord Rayleigh's model, shown in Figure 15$_b$, "two similar pulleys, A, B, turn upon a piece of round steel fixed horizontally. Over these is hung an endless cord, and the two bights carry similar pendent pulleys, C, D, from which again hang weights, E, F. . . . In the electrical analogy, the rotary velocity of A corresponds to a current in a primary circuit, that of B to a current in the secondary. . . . In the absence of friction there is nothing to correspond to electrical resistance, so that the conductors must be looked on as perfect. If x and y denote the circumferential velocities, in the same direction, of the pulleys A, B, where the cord is in contact with them, $\frac{1}{2}(x+y)$ is the vertical velocity of the pendent pulleys. Also $\frac{1}{2}(x-y)$ is the circumferential velocity of C, D, due to rotation, at the place where the cord engages. If the diameter be here $2a$, the angular velocity is $(x-y)/2a$. Thus, if M be the total mass of each pendent pulley and attachment, Mk^2, the moment of inertia of the revolving parts, the whole kinetic energy corresponding to each is

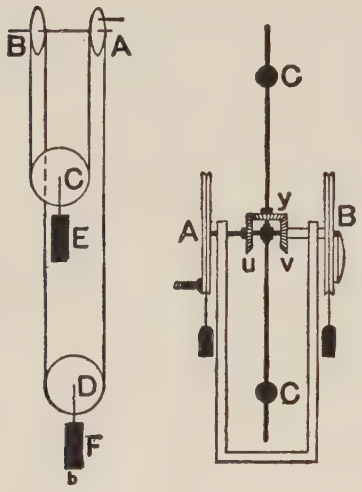


Figure 15

$$\tfrac{1}{2} M \left\{ \frac{(x+y)^2}{4} + \frac{k}{a^2}\left(\frac{(x-y)^2}{4}\right)\right\}. \qquad (25)$$

For the energy of the whole system, we should have the double of this, and, if it were necessary to include them, terms proportional to x^2 and y^2, to represent the energy of the fixed pulleys."

Here $L_1 = L_2 = a^2 + k^2$, $M = a^2 - k^2$, and, if there were no magnetic leakage, k would need to be zero.

Figure 16 represents the model of Professor Sir J. J. Thomson.[1] "It consists of three smooth, parallel, horizontal steel bars on which masses m_1, m_2, m slide, the masses being separated from the bars by friction wheels; the three masses are connected together by a light rigid bar which passes through holes in swivels fixed on the upper part of the masses; the bar can slide backwards and forwards through

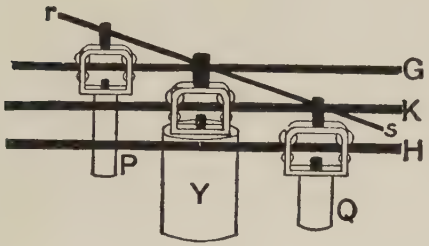


FIGURE 16

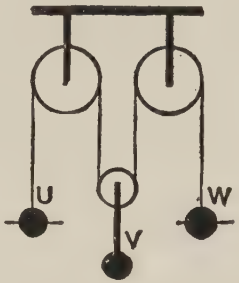


FIGURE 17

these holes, so that the only constraint imposed by the bar is to keep the masses in a straight line."

If x'_1, x'_2 are velocities of m_1, m_2, in the same direction, the velocity of M, if it be midway between m_1 and m_2 is $\frac{1}{2}$ $(x'_1 + x'_2$, and the kinetic energy is of the form

$$\tfrac{1}{2} L_1 x_1'^2 + M x'_1 \cdot x'_2 + \tfrac{1}{2} L_2 x_2'^2, \tag{26}$$

where $L_1 = m_1 + \frac{1}{4} m$, $L_2 = m_2 + \frac{1}{4} m$, $M = \frac{1}{4} m$.

Professor Webster's model is a modification of that of Thomson. "If the middle weight, instead of rolling on a fixed rail, roll on the bar connecting the two other carriages, the coefficients of induction will vary with the position of the middle mass, and moving it along its bar while one of the outer masses is moving will cause the other to move. The centrifugal force tending to make the middle mass roll along its bar will represent the magnetic forces between the currents."

The very elaborate and ingenious model of Boltzmann is described at length in the first fifty pages of his Vorlesungen über Maxwell's Theorie der Elektricität und des Lichtes.

[1] J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, chap. 11; Webster, *Science*, Dec. 1895; *The Theory of Electricity and Magnetism*, § 71.

The general features of another simple model illustrative of this electrical problem are shown in Figure 17. The mass of U is $L_1 - M$, that of V is $4M$, and that of W, $L_2 - M$. In Figure 18 the strings are represented as stretched over four small pulleys to keep them taut. I have found that this model made of three weighted roller skates, moving over a level table top, and connected in the manner indicated by cords passing around such small cheap pulleys as are obtainable at any ironmonger's shop, may be made to work extremely well. The effects of sudden changes of inductance can be directly observed by dropping suitable masses into the skates as they move. In Figure 14, which illustrates the same problem, the mass of Q is M, and those of P and Q are $L_1 - M$, $L_2 - M$, respectively. Q should have no vane.

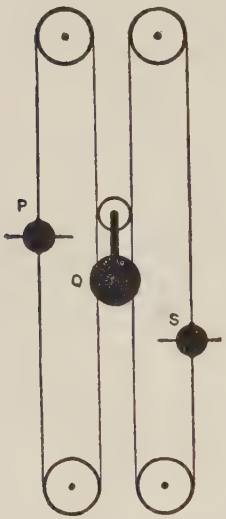


FIGURE 18

Scores of other models, more or less simple of construction, can easily be devised. It is to be noticed, however, that in some of the models which have been used to illustrate this problem, the masses representative of some of the combinations of the inductances would need to be negative if they were to correspond to cases which occasionally arise in electrical engineering.

If either of the two neighboring circuits contains no battery, the corresponding value of E in equations (24) is to be put equal to zero. Figure 19 [1] is drawn for the case of an induction coil without iron and with no cell in the secondary circuit. The self-inductances of the two circuits are equal. The dotted curve, P, shows the form of the current induced in the secondary circuit when the primary circuit, which has been carrying a steady current, is suddenly broken. If, after a few seconds, the primary circuit containing its battery be closed again, the current in the secondary circuit will have the general form of either Q or S. Q, R, and S are drawn for mutual inductances respectively half as great, nine tenths as great, and equal to, the inductance of either circuit. P is drawn for $M = L/2$, and corresponds, therefore,

[1] Identical with Fig. 2 on p. 103.

to Q; the areas V and W are equal. Curves like P corresponding to R and S could be found by exaggerating all of P's ordinates in the ratio 9/5, or the ratio 2.

Figures 20, 21, 22, 23, and 24 illustrate some phenomena which are frequently encountered in the practical use of neighboring inductive circuits. The curves have been drawn to scale for certain numerical values of the resistances, and the inductances so chosen as to make the results typical. There is no iron in either circuit, and only one circuit, the primary, contains a battery.

In Figure 20 the current in the primary circuit is drawn above OX, in the curve OJAZ, and the current in the secondary circuit beneath

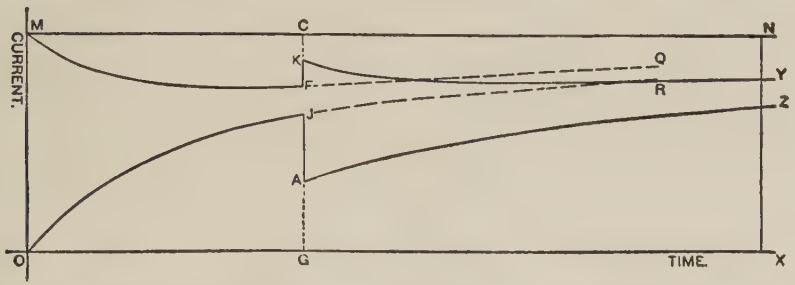


FIGURE 20.—OJR and MFQ represent the forms of the primary and secondary currents in a certain induction coil without iron when the primary circuit is closed at the origin of time. If at the time OG, the self-inductances of both circuits and the mutual inductance of the two are suddenly doubled, the currents take the forms OJAZ and MFKY.

MN. Each of the circuits has a self-inductance of 2 henries, and the resistances are 30 ohms and 40 ohms. The mutual inductance is at first $\sqrt{2}$, and the currents are given by the equations

$$C_1 = 4 - 2.4e^{-10t} - 1.6e^{-60t}, \quad C_2 = \frac{-2.4}{\sqrt{2}}(e^{-10t} - e^{-60t}), \qquad (27)$$

until the time OG = 1/20, when all the inductances are suddenly doubled. The currents are then given by the equations

$$C'_1 = 4 - 1.928\,e^{-5t} - 0.840\,e^{-30t}, \quad C'_2 = 0.891\,e^{-30t} - 1.361\,e^{-5t}. \qquad (28)$$

In the case represented by Figure 21, $r_1 = 30$, $r_2 = 40$, $M = \sqrt{2}$, $L_2 = 2$. At the beginning $L_1 = 2$, but at the time OA this is suddenly changed to 4. Before the change in L_1 the currents are given by the equations

$$C_1 = 4 - 2.4e^{-10t} - 1.6e^{-60t}, \quad C_2 = \frac{-2.4}{\sqrt{2}}(e^{-10t} - e^{-60t}). \quad (29)$$

Just before the impulse, $C_1 = 2.465$, and $C_2 = -0.945$; just after, the current in the primary is about 0.822 and the secondary current has

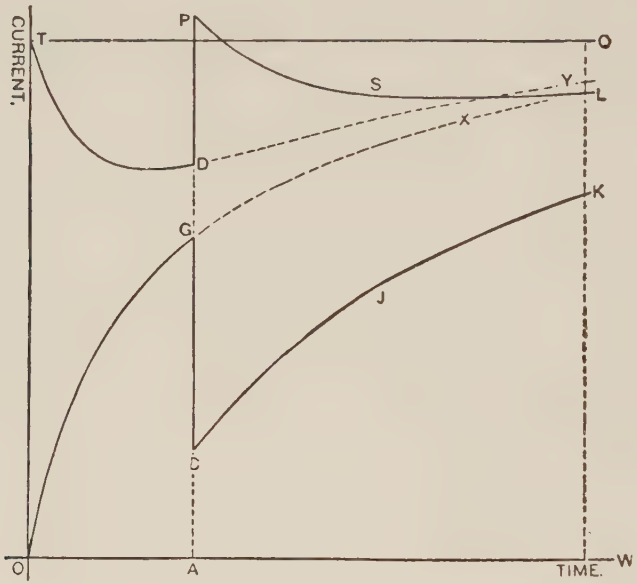


FIGURE 21. — At the time OA, the self-inductance of the primary circuit of a certain induction coil without iron is suddenly doubled, while the self-inductance of the secondary circuit and the mutual inductance of the two remain unchanged. OGCJK shows the course of the primary current and TDPSL that of the secondary current.

the small positive value 0.217. The new currents satisfy the equations

$$C'_1 = 4 - 2.634\, e^{-20t/3} - 0.543\, e^{-30t}, \; C'_2 = -0.932\, e^{-20t/3} + 1.149\, e^{-30t}, \quad (30)$$

very nearly. C_2 is plotted below TQ.

Figure 22 shows the manner of growth of two neighboring currents, when $r = 30$, $r = 40$, $L = 2$, $L = 2$, and when M, which is at first $\sqrt{2}$, is suddenly changed to zero at the time OA. When M is changed, the current in the primary circuit suddenly falls from 2.465 to 1.797, and the current in the secondary circuit, which has been negative,

rises from -0.945 to $+0.798$. After the change, the currents are given by the simple equations

$$C'_1 = 4 - 2.203\, e^{-15t}, \qquad C'_2 = 0.798\, e^{-20t}. \tag{31}$$

Figure 23 exhibits the effects of a sudden change in the value of the mutual inductance between the two circuits already described under Figures 21 and 22, from $\sqrt{2}$ to 1.9, while the other inductances remain

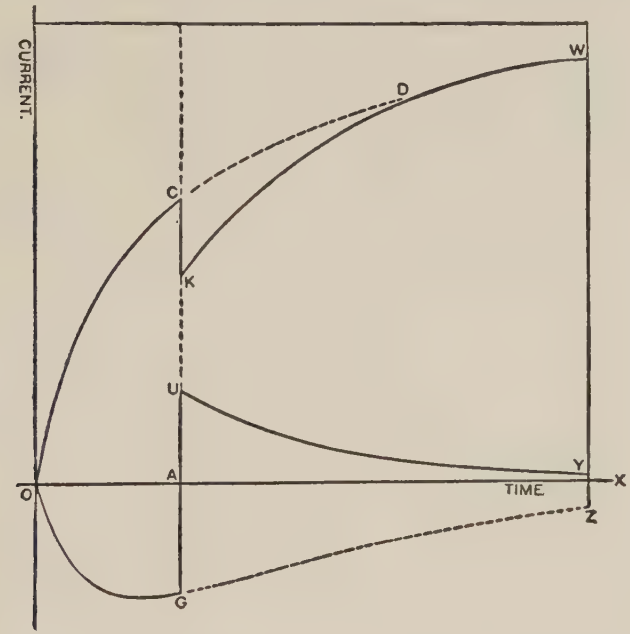


FIGURE 22. — At the time OA the mutual inductance of two neighboring circuits, one of which contains a battery, is suddenly reduced to zero. The primary and secondary currents which have been pursuing the courses OCD, OGZ are abruptly changed in value and now follow the lines OCKW and OGAUY.

unaltered. The primary current is shown by the curve OKRGS, and OX shows its final value. The second current is represented by the curve ADWU plotted under AN and displaced to the right so that the sudden increase in the absolute strength at the time of the change in M may appear. The flux of magnetic induction through the primary circuit is represented on the usual scale by the shaded area. The

black area points to a decrease in this flux which goes on from the time XF to the time XG, when the current falls below its final value. The induction flux linked with each of the two circuits is plotted against the time in Figure 24. These quantities are shown to be continuous at the time of change in the inductance, as, of course, they should be.

Figure 25 shows another arrangement of two neighboring circuits and an analogous mechanical system. The gap O is closed at first, but

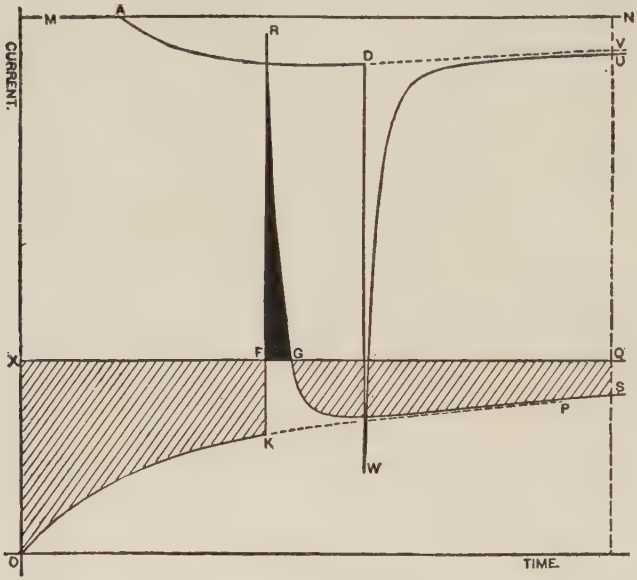


FIGURE 23.—The two circuits of a certain induction coil without iron have equal self-inductances (L, L) and the mutual inductance $L\sqrt{2}$. At the time XF the mutual inductance is suddenly increased to $(1.9)L$, and the currents which have been following the curves OKP, ADV, take the forms KRGS, DWU.

is suddenly opened when the current has become steady. The mass W moves alone under the action of a force E which urges it in the direction of the bottom of the page, and the air resistance. The motion soon becomes steady, but when the string which connects W to X becomes taut the motion is suddenly changed.

Figure 9_d represents a circuit consisting of three parallel branches each of which has self-inductance and may contain a battery, and

two of which have mutual inductance. If L, L_1, L_2 are the self-inductances, r, r_1, r_2 the resistances, C, C_1, C_2 the currents, E, E_1, E_2

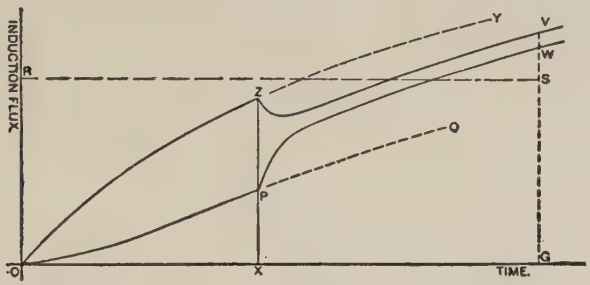


FIGURE 24. — OPW and OZV show, as functions of the time, the fluxes of magnetic induction linked with the two circuits which carry the currents represented in the last figure.

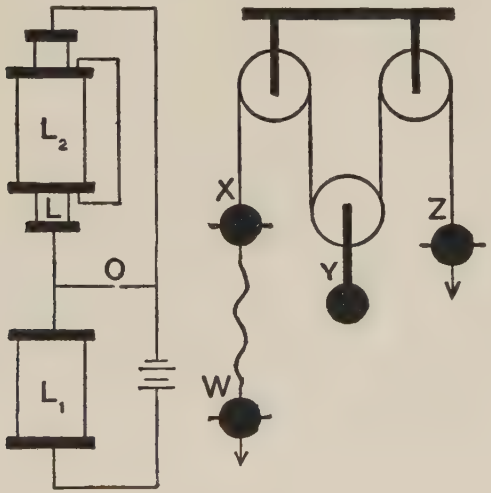


FIGURE 25

the electromotive forces, and M the mutual inductance of the second and third branches, the currents satisfy the equations

$$(L + L_1)\frac{dC_1}{dt} + (L + M)\frac{dC_2}{dt} + (r + r_1)C_1 + rC_2 = E + E_1,$$

$$(M + L)\frac{dC_1}{dt} + (L + L_2)\frac{dC_2}{dt} + rC_1 + (r + r_2)C_2 = E + E_2, \tag{32}$$

$$C = C_1 + C_2.$$

Any sudden changes in the inductances cause such sudden changes in the currents as shall keep $[(L + L_1) C_1 + (L + M) C_2]$ and $[(L + M) C_1 + (L + L_2) C_2]$ momentarily unchanged. In a case frequently met with in practice, there is no appreciable inductance in the first branch and no electromotive forces in the other branches, so that

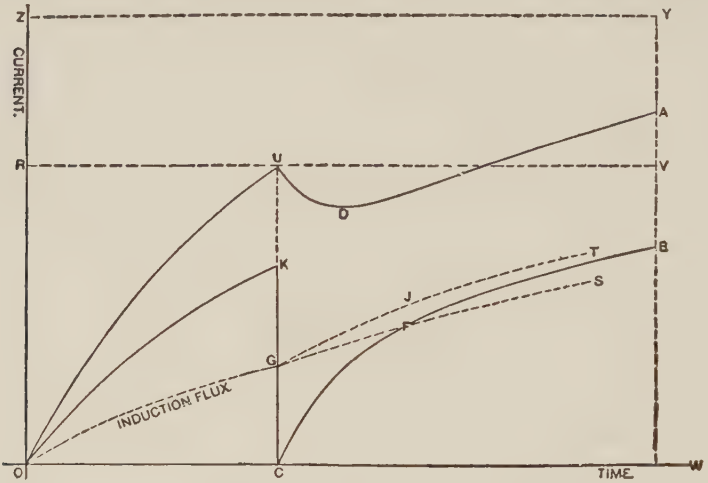


FIGURE 26. — Two parallel coils of resistances 20 ohms and 30 ohms and of inductances 2 and 3 respectively, connect, in parallel, the terminals of a storage battery of 1 ohm internal resistance. At the beginning there is no mutual inductance between the branch circuits and the currents follow the curves OU, OK. At the time OC, the conformation of the circuit is suddenly changed so as to introduce a mutual inductance of 2, and as a result, the courses of the two currents are altered: the first follows henceforth the line UDA, the second the line KCFB. The inductions linked with the parallel branches are shown by the dotted curves.

$L = 0$, $E_1 = 0$, $E_2 = 0$, and any instantaneous change in the inductances will leave $(L_1C_1 + MC_2)$ and $(MC_1 + L_2C_2)$ momentarily unchanged.

If in Figure 14 $P = L_1 - M$, $Q = L + M$, $R = L_2 - M$, and if the vanes are of such dimensions as to make the air resistance r_1, r, r_2 when the bodies to which they are attached have unit velocities, the equations of motion of the mechanical system are of the form (32), if E is applied upward to Q.

Figure 26 illustrates a special case under this problem where $r = 1$, $r_1 = 20$, $r_2 = 30$, $L_1 = 2$, $L_2 = 3$, $M = 0$, up to the time OC, when by

a sudden change in the conformation of the circuit, M is made equal to 2. Before the change $C_1 = 1.986$, $C_2 = 1.324$; the change in M leaves C_1 momentarily unchanged but suddenly reduces C_2 to zero. After the impulse the currents are given by the equations

$$C'_1 = 3 - 1.717\, e^{-5.959t} + 0.703\, e^{-54.541t},$$
$$C'_2 = 2 - 1.428\, e^{-5.959t} - 0.572\, e^{-54.541t}, \tag{33}$$

very nearly.

If in the arrangement shown in Figure 9_e, the gap O, which has been closed by a stout wire, is suddenly opened, the current falls impulsively to a value which keeps the induction flux through the battery circuit momentarily unchanged.

The mechanical system shown in Figure 27 is analogous to the electrical circuit indicated in Figure 9_f. The gap O, which has been closed, is supposed to be opened at a given signal. The spring S is the analogue of the condenser K.

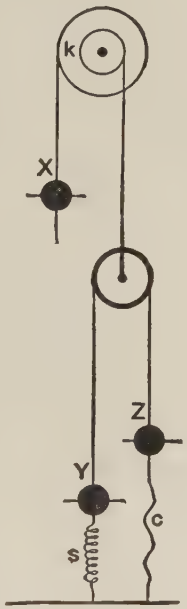


FIGURE 27

A circuit which contains an electromagnet has, of course, no definite inductance in the sense of the ratio of the flux of magnetic induction linked with the circuit to the intensity of the current, for this ratio is different for different current strengths, and for a given electromagnet, and a given current depends upon the previous magnetic history of the core.

In the case of a single circuit without iron the magnetic flux which accompanies a changing current is at every instant the same as it would be under a steady current of the intensity which the changing current then has. If, however, a second circuit closed on itself is brought into such a position that the two circuits have a mutual inductance, a changing current in the first circuit induces a current in the other which contributes to the flux through the first. If, therefore, an electromagnet has a solid core, the eddy currents induced in it while a current is growing or decreasing in the exciting coil affect the amount of the flux through the core, and it is

not possible to obtain a hysteresis diagram for the iron directly from the records of an oscillograph in the coil circuit. It is difficult, indeed, to obtain by any method a satisfactory magnetic curve for such a core, for if the iron starts from a given magnetic state, it is possible to get very different magnetic fluxes from a given exciting current by building up the current more or less slowly. In any useful examination of the magnetic properties of a solid piece of iron which is to

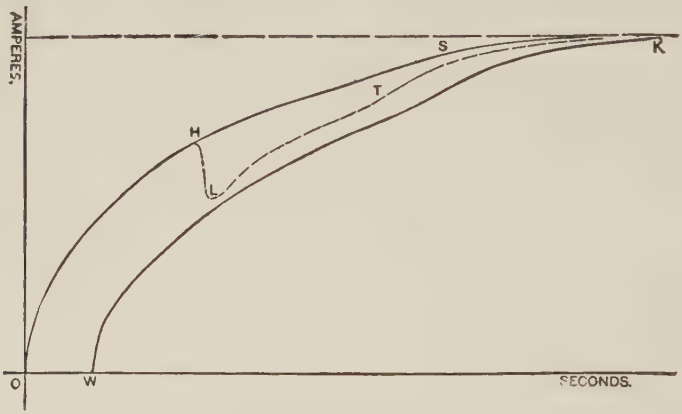


FIGURE 29. — The curve OHS shows the manner of growth of a current in the coil of the magnet TP when the poles are separated by about three inches. WK shows the rise of the same current when the poles are nearly closed by the insertion of a planed block of iron between them. HLT shows the effect of suddenly dropping the block in while the current is growing.

be used for any practical purpose, it is essential that the metal be made to go over the same magnetic journeys which it will later be required to make, and at the same speed.

Before we discuss this anomalous magnetization more carefully, we may stop for some moments to study the records of an oscillograph in circuits which contained either the electromagnet TP, Figure 28,[1] which has a solid core, or a certain toroid (DN), about 41 centimeters in mean diameter, the core of which was made of about 25 kilograms of fine, soft, varnished iron wire. It will be seen from Figures 29, 30, 31, 32, 33, 34 that the phenomena are in general what we should expect to find in similar circuits without iron, though eddy currents and the time taken to make the mechanical changes modify somewhat the courses of the currents in the exciting circuits.

[1] Identical with Fig. 1 on p. 211.

The Anomalous Magnetization of Iron

In 1863 von Waltenhofen first called attention to the fact that if an increasing current (C) ending in the maximum value (C') be sent through a long solenoid, the final value of the magnetic moment of a bar of soft iron in the solenoid, which was at the outset demagnetized, will depend not only upon the final strength of the current, but also upon the manner of growth of C in attaining this intensity. This

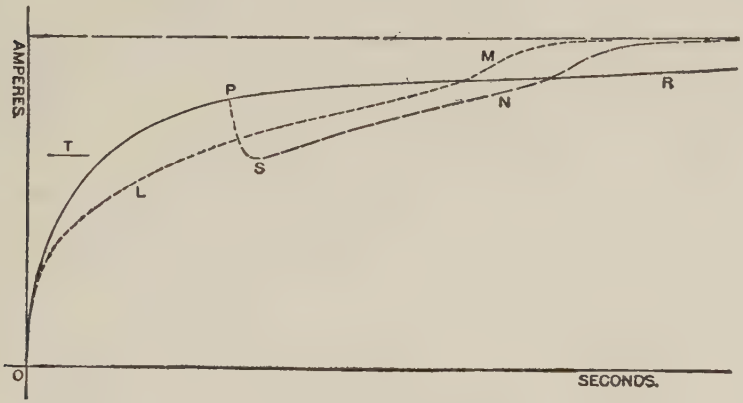


Figure 30. — The short coils of the magnet TP are in series with one another and with a battery and an oscillograph. The current follows the line OPR or the line OLM according as the long coil of the magnet is open or closed. When at the point P, the long coil circuit is suddenly closed, the current in the battery follows the line PSN, which despite eddy currents is not very unlike the upper part of OLM.

moment will be greater if the current be suddenly applied in full strength than if it be made to grow slowly, either continuously or by short steps. If, after the current has remained steady for a short time at the strength C' it be made to decrease to zero, the residual moment of the bar will be less if the circuit be suddenly opened than if the decrease be made slowly by introducing more and more resistance.[1]

[1] Von Waltenhofen, *Poggendorff's Ann.*, vol. cxx, 1863; Fromme, *Poggendorff's Ann.*, Ergbnd., vol. vii, 1876; *Wied. Ann.*, vol. iv, 1878, vol. v, 1878, vol. xiii, 1881, vol. xviii, 1883, vol. xliv, 1891; Bartoli and Alessandri, *Nuovo Cimento*, vol. viii, 1880; Righi, *Mem. di Bologna*, vol. i, 1880; Peuckert, *Wied. Ann.*, vol. xxxii, 1887; Auerbach, *Wied. Ann.*, vol. xiv, 1881, vol. xvi, 1882; Winkelmann's *Handbuch der Physik*, Band v; Wiedemann, *Lehre von der Elektricität*, Band iv; Ewing, *Magnetic Induction*, § 84; Gumlich and Schmidt, *Electrotechnische Zeitschrift*, vol. xxi, 1905.

If the soft iron bar to be magnetized was stout and relatively short, von Waltenhofen was sometimes able to reverse the direction of the remanent magnetism by a sudden break of the circuit. In one instance where the length of the bar was about ten centimeters and the diameter about two centimeters, the magnetic moment while the current was passing was about 45 units, and about -0.20 when the current had been stopped. It seemed to von Waltenhofen that these phenomena could not be due to the induced currents caused by the sudden

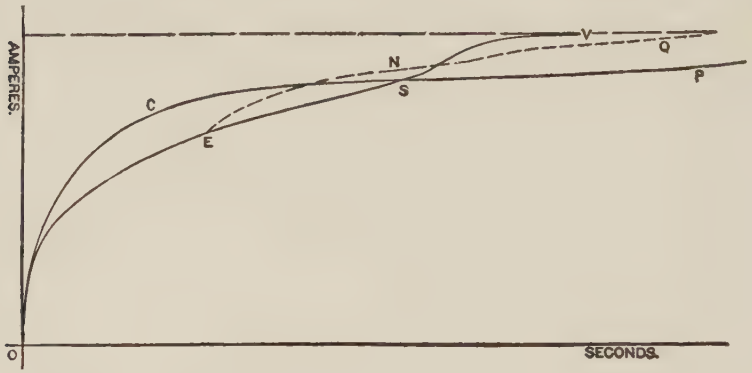


FIGURE 31. — The short coils of the magnet TP are in series with one another and with a battery and an oscillograph. The current follows the line OCP or OESV according as the long coil of the magnet is open or closed on itself. When at the point E the long coil which has been closed is suddenly opened, the oscillogram is of the form ENQ which on account of the disturbing effects of eddy currents is not like the upper part of the curve OCSP.

changes in the exciting current, and he explained them as consequences of the inertia of the molecular magnets turning in a viscous medium. This view seems to have been taken by Fromme, Auerbach, Ewing, Peuckert, Zielinski and others who have written upon the subject, while G. Wiedemann thought that his researches and those of Righi showed that eddy currents in the iron, and alternating currents induced in the exciting coil accounted best for the observed facts. In describing experiments upon this so-called anomalous magnetization, Wiedemann distinguishes between the permanent moment (P), that is, the remanent moment after the current has ceased, and the total moment (T), which is the moment when the current is steady at its highest value. This last quantity is regarded as the

sum of the permanent moment and a moment (V) which vanishes with the current. The suffix a attached to P or T denotes that this moment has been reached after a gradual change in the current, while the suffix f denotes that the current has been suddenly opened or closed. According to Wiedemann, T_a is always smaller than T_f, and P_f than P_a, but these differences are much larger for short stout

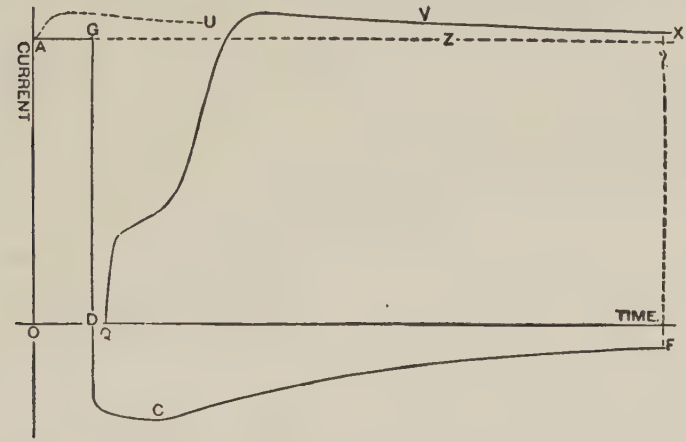


FIGURE 32. — The toroid (DN) and the electromagnet (TP) with its poles separated by a gap of about eight centimeters, were placed in parallel across the terminals of a storage battery, with an oscillograph in the toroid branch. QVX shows the course of the current in the toroid which approaches its final value from above (see Figure 12). GCF, with its irregularities of curvature, shows the current in the toroid when the battery circuit was suddenly broken. AU shows the form of the temporary current induced in the toroid when an iron block was suddenly dropped into the gap between the jaws of the electromagnet, after the currents in the circuit had become steady.

rods than for relatively long ones, where they become insignificant. $(P_a - P_f)/P_a$ is smaller in the case of a rod made of a bundle of insulated soft iron wires than in the case of a solid rod of the same dimensions. As C' is made larger, $T_f - T_a$ attains a maximum and then sometimes decreases slightly. If the rod to be magnetized is surrounded by a thick metal tube in which eddy currents can be induced, P_f is slightly increased, especially if the current be first slowly raised to C and then suddenly stopped. If shorter and shorter iron rods of a given diameter are tested, P_f gradually decreases to zero under given value of C', and then changes sign; the inversion comes

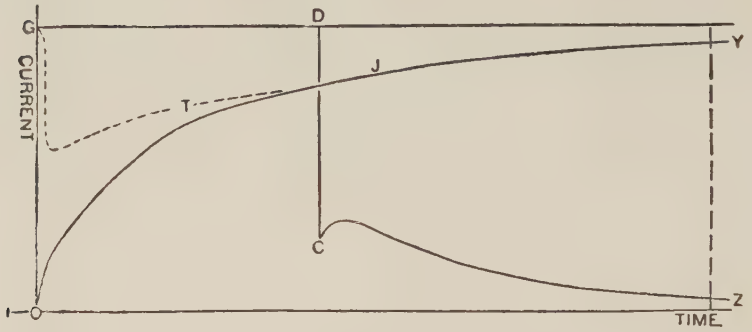


FIGURE 33. — The toroid (DN) and the electromagnet (TP), with its jaws separated by a gap of about eight centimeters, were placed in parallel across the poles of a storage battery, with an oscillograph in the electromagnet branch. OJY shows the manner of growth of the current in TP and DCZ the manner of decay of the current when the battery circuit was suddenly broken. If, after the currents in the circuit have become steady, a block of iron was suddenly dropped into the gap in the core of the electromagnet, the induced current took the form GT.

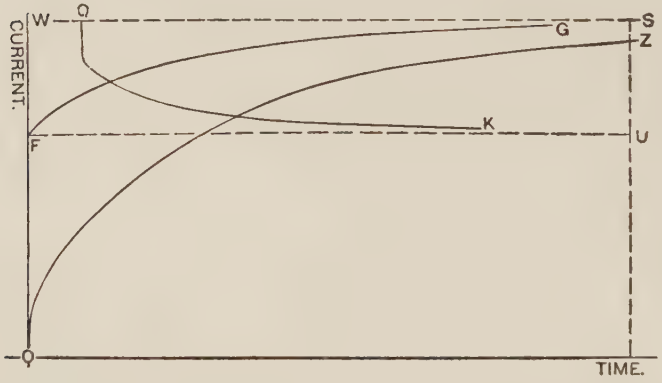


FIGURE 34. — Two electromagnets were placed in series with each other and with an oscillograph and a storage battery, and a shunt (S) of small resistance was provided for one of the magnets. OZ shows the form of the battery current when S was closed, QK the fall of this current when S was suddenly opened after the original current had become steady, and FG the rise of the current to its old value when the shunt is again closed.

with longer rods when C' is weak than when it is strong. If, with a given rod, C' be gradually increased, the negative moment finally decreases and changes sign. After this there is no inversion and P_f is positive.

It appears from Fromme's experiments that the von Waltenhofen effect is often less marked in straight, finely divided cores than in solid ones, and we may inquire how greatly the division of a straight core may be expected to facilitate the changes in the field (H) within the iron, due to given changes in the exciting circuit. It is clear that if the circuit of an electromagnet be suddenly broken, the decay of the electromagnetic field in the core is much less rapid when the core is solid and eddy currents induced in it shield the inner filaments, than when it is made of wire. Indeed, if eddy currents were non-existent, the field would fall instantaneously to zero, in the absence of magnetic lag, when the current in the coil ceased to flow. If the exciting coil remains closed and some change is suddenly made in its resistance or in the electromotive force applied to it, the change in the current and therefore the change in the field in the iron caused by the current cannot be made instantaneous, even if eddy currents be wholly shut out, and, though dividing up the core has its effects, we cannot expect them to be so striking as in the case where the exciting circuit is open.

Let us consider a very long, uniformly wound solenoid consisting of N turns of insulated copper wire per centimeter of its length, wound closely upon a long, soft iron prism of square cross-section ($2a \times 2a$) built up compactly of a large number of straight, varnished filaments or "wires" of square cross-section ($c \times c$), with their axes parallel to that of the prism, which shall be used as the z axis. The electric resistance of the solenoid per centimeter of its length parallel to the z axis shall be w, the constant applied electromotive force per centimeter of the axis shall be E, and the intensity of the current in the coil shall be C.[1] Within the core, the magnetic field (H) will have everywhere and always the direction of the axis of the prism, and if q is the current flux at any instant at any point in the iron, ρ the specific resistance of the metal, and μ its magnetic permeability, which for the present purpose shall be regarded as having a fixed uniform value, $q_x = 0$, $q_y = 0$, $H_x = 0$, $H_y = 0$, $H_z = H$, $4\pi q = \text{Curl } H$,

[1] Peirce, *Proceedings of the American Academy*, vol. xliii, 5, 1907.

$$\frac{\partial H}{\partial t} = \frac{\rho}{4\pi\mu}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y}\right). \tag{34}$$

When there are no eddy currents in the core, the intensity (H) of the magnetic field has at every point of the iron the boundary value $H_s = 4\pi NC$, but in general H varies from point to point. The flux of magnetic induction through the turns of the coil per centimeter of its length parallel to the z axis and N times the induction flux through the core are practically equal, and we may write

$$E - \frac{dp}{dt} = E - \mu N \iint \frac{\partial H}{\partial t} \cdot dxdy = w \cdot C = \frac{w \cdot H_s}{4\pi N}, \tag{35}$$

or,

$$E = \frac{w \cdot H_s}{4\pi N} + \frac{\mu\rho N}{4\pi\mu} \iint \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) dxdy, \tag{36}$$

where the integration extends over a cross-section of the core.

The vector H is always perpendicular to its curl, and the intensity of the component of the current at any point in the iron, in any direction s, parallel to the xy plane at any instant, is equal to $1/4\pi$ times the value at that point, at that instant, of the derivative of H in a direction parallel to the xy plane, and 90° in counter clockwise rotation ahead of s.

Along any curve in the iron parallel to the xy plane, H must be constant if there is no flow of electricity across the curve. At every instant, therefore, the value of H at the boundary common to any two filaments must be everywhere equal to H_s. If the coil circuit is broken, H must be constantly zero at the surface of every filament.

$$E = \frac{w \cdot H_s}{4\pi N} + \frac{\mu\rho n^2 N}{4\pi\mu} \iint \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) dxdy, \tag{37}$$

where the integral is to be taken over the section of one filament.

Eddy currents in any filament which are the same in direction and intensity at all points on any line parallel to the axis do not affect in any way the magnetic field outside the filament.

If, after a steady current E/w has been running in the solenoid, the circuit be instantaneously broken, the value of H falls from $4\pi NE/w$ to 0 at the surface of the prism and at the surface of every filament, and H is given by the equation

$$H = \frac{16\,H_0}{\pi^2} \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=\infty} \frac{e^{-\lambda^2 t}}{(2j+1)\,(2k+1)} \cdot \sin \frac{(2k+1)\,\pi x}{c} \cdot \sin \frac{(2j+1)\,\pi y}{c}, \tag{38}$$

$$\lambda^2 = \frac{\pi \rho}{4\mu c^2} [(2k+1)^2 + (2j+1)^2], \tag{39}$$

in every filament. The origin is at one corner of the cross-section of the filament, and the x and y axes are two sides of the section; k and j are integers, and $H_0 = 4\pi NE/w$.

If we differentiate both members of equation (38) with respect to the time, and integrate the result over the cross-section of a filament, we get for the average value for the whole core, at the time t, of the rate of change of the magnetic force in the iron,

$$\frac{1}{c^2} \int\int \frac{\partial H}{\partial t} \cdot dS = \frac{4\,\rho H_0}{\mu \pi^3 c^2} \sum_{j=0}^{j=\infty} \sum_{k=0}^{k=\infty} \frac{[(2k+1)^2 + (2j+1)^2]\, e^{-\lambda^2 t}}{(2j+1)^2\,(2k+1)^2} = \frac{4\rho H_0}{\mu \pi^3 c^2} \cdot M. \tag{40}$$

At the origin of time, M has the same maximum value for all values of c; and for different values of c (c', c''), M has the same numerical value at times (t', t'') such that $t'/c'^2 = t''/c''^2$, provided μ has the same value in both cases. The smaller the values of μ and c the sooner does M for a given core attain a given value. At $t = 0$, when the change of H in the iron is most rapid at all points, the average value of dH/dt throughout the core is inversely proportional to c^2, that is, the average rate at which H is changing is 100 times as great when the core consists of filaments only one millimeter square as when the filaments are a centimenter square. This analysis illustrates the fact that when the main circuit of an electromagnet is suddenly broken, the changes in excitation to which the iron in a divided core is subject are far more violent than those which the particles of a solid core encounter. It is to be noticed that the average value of dH/dt given above is proportional to the electrical conductivity of the iron and to the intensity of the steady current in the exciting coil before the break.

In a similar manner, it is possible to show that if a given current in the exciting coil of an electromagnet be changed by a sudden increase or decrease in the resistance of the circuit while the applied electromotive force is unaltered, the whole given change of the magnetizing field in the core takes place somewhat more quickly if the core is finely

divided than if it is solid. The general fact is, of course, evident without computation.

If the square core of a solenoid, the area of the cross-section of which is A square centimeters, be made of a bundle of infinitely long, straight iron wires placed close together, and if after a steady current of intensity E/w has been running for some time through the circuit so that there is a magnetic field of uniform intensity $H_0 = 4\pi NE/w$ in the core, the resistance of the solenoid circuit be suddenly changed to w' ohms per centimeter of length of the core, the current in the coil will gradually change to E/w', and the field in the core finally reaches the uniform value $H_\infty = 4\pi NE/w'$. At any instant the field in so much of the space A as is occupied by air is $4\pi NC$, for eddy currents in the round wires act like solenoidal current sheets, and do not affect the field outside the wires. Within each wire there are, of course, eddy currents, and at every point in the iron at every instant, the field intensity, H, must satisfy the equation (34).

The induction flux (p) through the solenoid per centimeter of its length is

$$4\pi N^2 C\,(A - n^2 B) + \mu N \iint H \cdot dxdy, \tag{41}$$

where n^2 is the number of wires in the core and B is the area of the cross-section of each of them. The double integral is to be extended over the cross-sections of all the wires.

Since
$$E - \frac{dp}{dt} = w'C,$$

$$w'C + (A - n^2 B)\,4\pi N^2 \cdot \frac{dC}{dt} + \mu N \iint \frac{\partial H}{\partial t} \cdot dS = E, \tag{42}$$

and if H_s represents the strength of the magnetic field in the air space within the solenoid, and $A - n^2 B$ is written $h \cdot A$,

$$H_S - \frac{4\pi NE}{w'} + \frac{4\pi N^2 hA}{w'} \cdot \frac{dH_S}{dt} + \frac{4\pi\mu n^2 N^2}{w'} \iint \frac{\partial H}{\partial t} \cdot dS = 0, \tag{43}$$

where the double integral is to be taken over the cross-section of a single filament. If we put $H = H' + H_\infty$ and $H_s = H'_s + H_\infty$, the last equation becomes

$$H'_S + \frac{4\pi N^2 hA}{w'} \cdot \frac{dH'_S}{dt} + \frac{4\pi\mu n^2 N^2}{w'} \iint \frac{\partial H'}{\partial t} \cdot dS, \qquad (44)$$

in which H' satisfies at every point the equation

$$\frac{\partial H'}{\partial t} = \frac{\rho}{4\pi\mu}\left(\frac{\partial^2 H'}{\partial x^2} + \frac{\partial^2 H'}{\partial y^2}\right) = \frac{\rho}{4\mu\pi}\left\{\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial H'}{\partial r}\right)\right\} \qquad (45)$$

where r is the distance from the axis of the wire in which the point lies. We are to find a function H' which satisfies equations (44), (45), which, when $t \equiv 0$, is everywhere equal to $H_0 - H_\infty$, and which vanishes everywhere when t is infinite

If $$\bar{\omega} = \sum L e^{-\beta^2 t} \cdot J_0(mr), \qquad (46)$$

in which either m or β may be chosen at pleasure and the other computed from the equation

$$m^2\rho = 4\pi\mu\beta^2, \qquad (47)$$

and if for m in (46) we use the successive roots of the transcendental equation

$$J_0(mb)\left(1 - \frac{N^2 hA\rho \cdot m^2 b^2}{\mu w' b^2}\right) = \frac{2\pi n^2 N^2 \rho}{w'} \cdot mb \cdot J_1(mb), \qquad (48)$$

where b is the radius of the wire, $\bar{\omega}$ satisfies equations (44), (45) and vanishes when t is infinite.

Without any consideration of the question of a possible development of unity in terms of an infinite series of Bessel's Functions of the form $J_0(mr)$ where the m's have the values just mentioned, it is clear [1] that, within the comparatively short range from 0 to b, unity may be represented with sufficient accuracy by a few terms (sometimes two) of the form

$$L_1 \cdot J_0(m_1 r) + L_2 \cdot J_0(m_2 r) + L_3 \cdot J_0(m_3 r) + \ldots = \sum L_k \cdot J_0(m_k r), \qquad (49)$$

so that $$H = H_\infty + (H_0 - H_\infty) \sum L_k \cdot e^{-\beta^2 t} \cdot J_0(m_k r) \qquad (50)$$

gives the value of the magnetic field at the time t at any desired point in the wire in question, and, therefore, at any desired point in any other wire of the core.

[1] Byerly, *Annals of Mathematics* for April, 1911.

$$\frac{\partial H}{\partial t} = \frac{-(H_0 - H_\infty)\,\rho}{4\pi\mu} \sum L_k \cdot m^2 e^{-\beta^2 t} \cdot J_0(mr), \tag{51}$$

and if this be integrated over the cross-section of a wire and divided by πb^2, the result,

$$\Omega = \frac{-(H_0 - H_\infty)\,\rho}{2\pi\mu b^2} \sum L_k \cdot e^{-\beta^2 t} \cdot mb \cdot J_1(mb), \tag{52}$$

will represent the average value in the whole core, at the time t, of the time rate of change of the magnetic field. An example will best show the meaning of these rather intractable expressions.

Suppose the core of a long solenoid of square cross-section, ten centimeters on a side, to be built up of straight, round iron rods one millimeter in diameter placed close together; then $h = 0.2146$, $b = 0.05$, $n = 100$. If the resistance of the solenoid coil per centimeter of its length is $\frac{1}{16}$ of an ohm, the specific resistance of the iron 9950 absohms, the number of turns of wire per centimeter of the solenoid 10, and the value of the permeability of the iron 100, then $mb = x$ satisfies the equation

$$J_0(x) \cdot (1 - 1.3666\,x^2) = 1000\,x \cdot J_1(x), \tag{53}$$

and the first root $x = 0.04465$ will suffice, for $m = 0.8930$; and $J_0(0.8930\,r)$ differs from unity by less than one tenth of one per cent over the whole range from $r = 0$ to $r = b$, and from (50)

$$\frac{H - H_\infty}{H_0 - H_\infty} = e^{-6.315t} \cdot J_0(mr), \tag{54}$$

very approximately. In the case of a core of the same cross-sectional area $(0.7854A)$, and the same permeability, but wholly without eddy currents, it is easy to show that

$$H = H_\infty + (H_0 - H_\infty) \cdot e^{-kt}, \tag{55}$$

where $k = w't/4\pi N^2 \cdot D, \qquad D = A\,[4 + \pi(\mu - 1)]/4.$

For this problem, (55) yields

$$\frac{H - H_\infty}{H_0 - H_\infty} = e^{-6.316t}, \tag{56}$$

and a comparison of (54) and (56) shows that the eddy currents in a core of this wire, one millimeter in diameter, have practically no effect in slowing the changes in magnetism of the iron.

If the core of the given solenoid were made up of rods one centimeter in diameter, mb or x would be given as the roots of the equation

$$J_0(x) \cdot (1 - 0.01366\, x^2) = 10\, x \cdot J_1(x), \qquad (57)$$

and it is not very difficult to show by a process of trial and error from Meissel's *Tafel der Bessel'schen Functionen*, that the first three of these roots are approximately equal to 0.4411, 3.8525, 7.0204, and that the corresponding values of $J_0(x)$ and $J_1(x)$ are 0.951946, -0.402672, 0.300112, and 0.215229, -0.008352, 0.001444.

If, with these roots, we wish to determine such a set of coefficients (L_1, L_2, L_3), as shall make the mean square of the difference between unity and $\Sigma L \cdot J_0(mr)$ as small as possible, for the range from $r = 0$ to $r = b$, we have to solve the equations

$$A_1 \cdot L_1 + B_{12} \cdot L_2 + B_{13} \cdot L_3 = C_1, \qquad B_{12} \cdot L_1 + A_2 \cdot L_2 + B_{23} \cdot L_3 = C_2$$
$$B_{13} \cdot L_1 + B_{23} \cdot L_2 + A_3 \cdot L_3 = C_3, \qquad \text{where}$$
$$C_1 = 2\pi b^2 \cdot J_1(x_1)/x_1, \qquad A_1 = \pi b^2 \{[J_0(x_1)]^2 + [J_1(x_1)]^2\},$$
$$B_{12} = b^2 [x_1 \cdot J_0(x_2) \cdot J_1(x_1) - x_2 \cdot J_0(x_1) \cdot J_1(x_2)]/(x_1^2 - x_2^2),$$

as Professor Byerly's theorems show. The computation here indicated shows that $\Omega = 6.096$ or 0.320, approximately, according as $t = 0$ or $t = 0.1$, whereas, if eddy currents were wholly cut out, the corresponding values would be 6.316 and 0.336. These figures illustrate the comparatively slight effect of subdividing the core in the particular case here considered. The results would, of course, be somewhat different numerically, with different assumed values for the constants of the circuit.

It is clear that the inversion of sign in the magnetic moment of a straight iron bar, when the magnetic excitation is suddenly removed, accompanies, at least, a large demagnetizing factor due to the ends of the bar, and no one seems to have observed the phenomenon in the case of closed cores. In rings, however, as in straight bars, the ultimate value of the intensity (I) of magnetization depends very much upon the manner in which the given exciting current is made to attain its final strength.

The experiments of Rücker [1] upon small solid iron toroids seem to show that at moderate excitations there may be a difference of from

[1] Babbitt, *Proceedings of the American Academy*, vol. xlvi, 1911; Rücker, *Inaugural Dissertation*, Halle-Wittenberg, 1905.

6 per cent to as much as 30 per cent in the final flux density due to a given current, according as the current is applied suddenly or by many short steps, and, unlike some other observers, he found a very real difference ($T_f - T_a$), though a smaller one than in the case of the solid metal, for a toroid with core of fine iron wire (Blumendraht). In the case of a large electromagnet with solid, closed core, weighing altogether more than 1500 kilograms, Babbitt found, by a very ingenious method of procedure, a difference of 17.4 per cent between the final flux density in the iron caused by the sudden application of a given current, and the growth from nothing of the same current in 56 steps. The cross-section of this massive core is more than 450 square centimeters in its narrowest part, and eddy currents are so much in evidence that quite two minutes are required for a "suddenly applied" current to attain its steady value.

Babbitt also carried out a long series of very accurate measurements extended over several months, upon two small toroids of fine, carefully annealed iron wire, and upon a toroid weighing more than 40 kilograms made of very well softened iron wire about half a millimeter in diameter. His results show conclusively that if one of these softened and demagnetized cores has been first put through the cycle due to a given excitation a considerable number of times to obliterate the effects of the past experiences of the iron, the form of the hysteresis diagram is precisely the same, whether the half cycle be carried out by one reversal of the main switch or in a very large number of steps. In general agreement with these results are some less accurate ones which I obtained three years ago in experimenting upon a transformer which has an exciting coil of 1394 turns and a core of about 120 square centimeters in cross-section, built up of thin strips of varnished sheet iron about ten centimeters wide. This transformer was connected in simple circuit with a storage battery and a rheostat besides a suitable oscillograph. When the circuit was suddenly closed, with such a resistance (x) in the rheostat that the final strength of the current was about 1.10 amperes, the current curve was of the form R as shown in Figure 35,[1] and when after a few seconds x was suddenly removed, so as to bring the final strength of the current up to about 2.30 amperes, the current curve was Q. When the whole journey was made without

[1] Identical with Fig. 30 on p. 135.

x the current curve was T. The sum of the flux changes represented by the shaded areas as measured by a Coradi "Grand Planimètre Roulant et à Sphère" was 1126, while the flux change corresponding to the area above the curve T was 11.30. The core was not sufficiently well divided to avoid all evidence of eddy currents, for the curve Q does not exactly conform throughout with the upper part of T. This is shown more clearly in Figure 36,[1] taken with the same transformer. Here the area of the shaded portion above K multiplied by the resistance then in the circuit should be equal to so much of the area above C, multiplied by the resistance belonging to it, as lies to the left of the dotted line which rises at about 1.1 seconds after the circuit was closed, and is an exact copy of the curve D moved to the left. This curve coincides with C for a large part of its course, but has a trifle less area above it than that portion of C has which lies to the right of the ordinate at which the lowest part of the dotted curve begins. The shape of D just at the beginning points to the existence of eddy currents.

To test more thoroughly the effect upon the flux of magnetic induction through the core of the transformer, of building up the current in different ways, I first measured with great care, by aid of a modified Rubens-du Bois "Panzer Galvanometer," the changes of this flux for a quick reversal of an excitation of 1812 ampere turns. I then measured by means of the planimeter a long series of oscillograph records obtained by reversing the same excitation by a considerable number of steps. All the testing instruments were different in the two cases, and no comparison was possible until the final results were reached and were found to differ from one another by only one part in fourteen hundred. The labor of reducing the oscillograms was so great that this close agreement must be considered accidental, but there can be little doubt, I think, that the flux change due to the single step and the sum of the changes due to the long series of steps which together cover the same change of excitation were practically indistinguishable.

Figure 37[1] shows copies of oscillograms taken with a number of toroids in series. The core of each toroid was made of perhaps fifteen kilograms of very soft, varnished iron wire, about one tenth of a milli-

[1] Figs. 36 and 37 are identical with Figs. 53 and 52 on pp. 159 and 158 respectively.

meter in diameter. The curves OHD, PDXU were taken when the cores had been thoroughly demagnetized just before the experiment; the curves MNC, QCZB after the core had been put a number of times

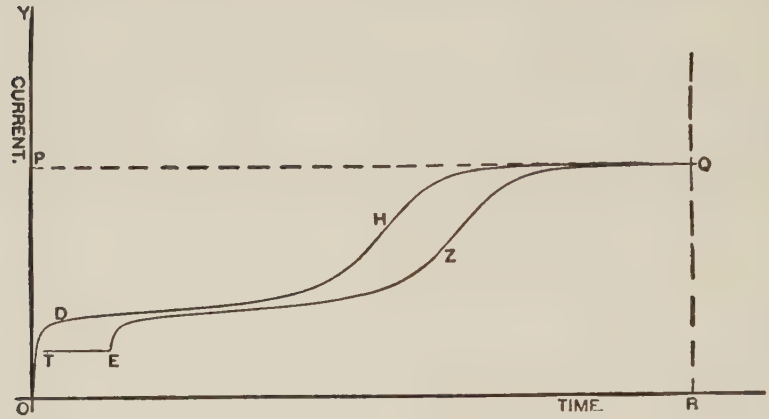


FIGURE 38. — ODHQ represents the curve of growth of a current in the exciting coil of the toroid DN, if the circuit was suddenly closed when its resistance was r. If the circuit was first closed with a higher resistance $(r + s)$, which corresponded to a steady current of intensity OT, and if the resistance s was suddenly shunted out, the current rose to the intensity OP in the manner indicated by the curve EZQ, which, as Figure A shows, is of exactly the same form as the upper portion of ODHQ.

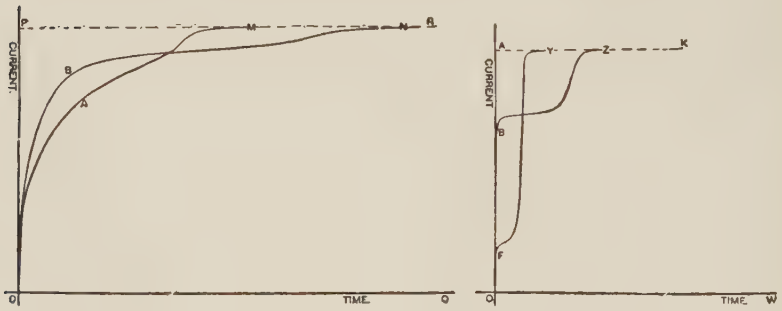


FIGURE 39

through the cycle corresponding to the excitation used. The toroids were in simple circuit with a storage battery, an oscillograph, and a rheostat of resistance x; when the circuit was suddenly closed the current grew to the final value corresponding to this resistance by the curve OHD or MNC, as the case might be. When at the proper time

the rheostat resistance was suddenly shunted out of the circuit, the current rose to the value OA by the curve DXU or the curve CZB. If x had been shunted out at the start the current curve had the shape accurately represented, when the starting point had been shifted just far enough to the right, by PDXU or QCZB. It was not possible to detect any difference between the curves DXU and CZB and the upper parts of the curves obtained with x all the time out of circuit.

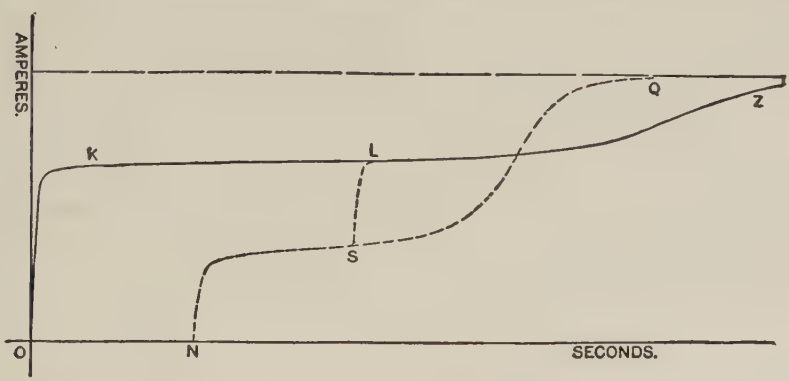


FIGURE 40. — The current in one coil of the toroid DN which is in series with a battery follows the line OKLZ or the line NSQ, according as another coil on the same core is open or closed. When this last coil which has been closed is suddenly opened while the battery current is rising, this changes abruptly and follows exactly the upper part of the line OKLZ.

This figure was drawn by superposing several oscillograms, for it is very difficult after one curve has been taken upon the sensitized paper carried by the revolving drum to start another curve some time afterwards at such a point that it shall coincide with the upper part of the first one. This feat has, however, just been accomplished in another case by Mr. John Coulson, who made the records shown in Figures A and B, and has helped me in most of the experimental work of this paper. Figure 38, drawn from another photograph, shows the two curves which coincide in A. The oscillograph was in circuit with the coil of a large toroid of about 41 centimeters in diameter, the core of which is made of soft, varnished iron wire about half a millimeter in diameter. Each record shows a current curve obtained by applying the electromotive force directly to the circuit, and the second part of a current diagram when an extra resistance, at first in the circuit, was suddenly shunted out. There seems to be in these cases neither

a magnetic time lag nor any sensible von Waltenhofen effect.

If an electromagnet has two exciting coils, and if one of them be attached to the terminals of a battery, the form of the battery current will depend upon whether the second coil is open or closed on itself, and the difference is usually noticeable even when the magnet has a large solid core in which eddy currents are being induced. Figure 39 shows curves taken under the two conditions just mentioned for both the electromagnet TP and the toroid DN. To determine whether the closing of the second coil in the case of the electromagnet where strong

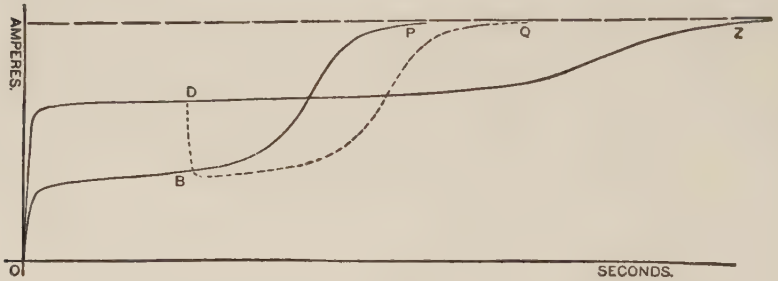
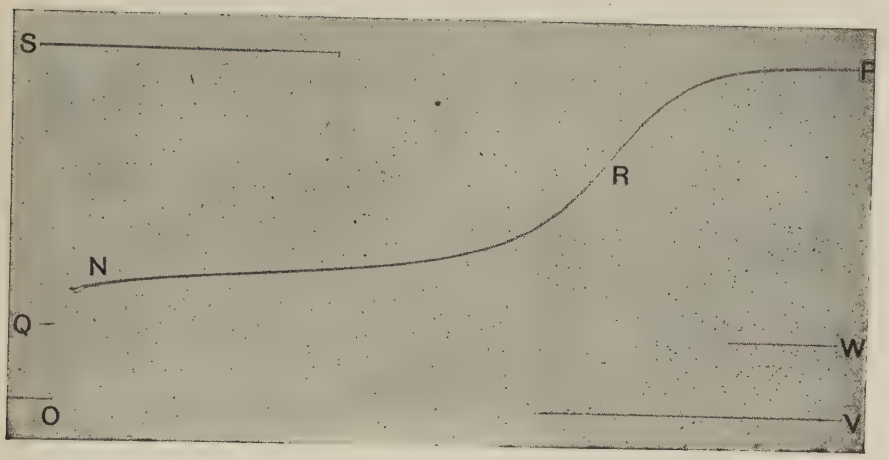


FIGURE 41. — The current in one coil of the toroid DN which is in series with an oscillograph and a battery, follows the curve ODZ or the curve OBP, according as another coil wound on the same core is open or closed in itself. When at the point D, the second coil which has been open is suddenly closed, the oscillograph record gives the curve DBQ which except at the very beginning can be exactly superposed upon the upper part of the line OBP.

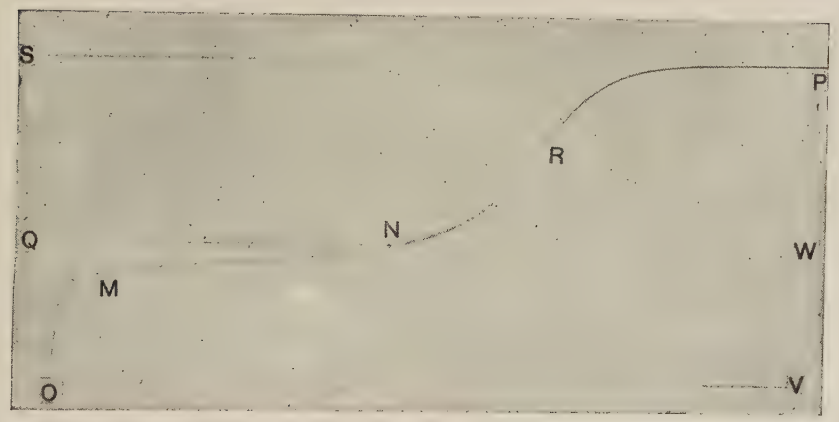
eddy currents already existed changed the amount of the final flux through the circuit, Mr. Coulson has measured with great care a number of oscillograms taken with this apparatus, and finds the area between the asymptote and the curve OAM to be 6216 on the scale of his planimeter, while the area above the curve OBN is 6214 on the same scale. The areas above the curves agree within a small fraction of one per cent, as they were expected to do.

Figures 40 and 41 exhibit oscillograms taken with the toroid, DN, under sudden opening and closing of the second coil, and these show no signs of von Waltenhofen effects. Figure 42[1] gives the records of two oscillographs, one in the primary circuit of a toroid which has a core made of soft iron wire only one tenth of a millimeter in diameter,

[1] Identical with Fig. 56 on p. 164.



A



B

the other in a secondary coil, when a third coil, wound on the same core, was suddenly closed.

In early experiments upon the phenomenon of the reversal of moment in short rods magnetized in a solenoid, when the current was suddenly stopped, it was observed that if the rod had been previously magnetized permanently in the direction in which the current magnetized it, reversal never occurred, but that it always appeared, under favorable circumstances, if the direction of the previous magnetization was the opposite of that which the current gave it. This and like results has led many physicists to think that the molecules of the iron, when the exciting force due to the current is suddenly removed, return to the positions which they had just before the current acted upon them, but that the motion is so much resisted by frictional forces that the kinetic energy is lost when the particles have swung slightly beyond the positions of equilibrium where they are held by the friction. Wiedemann believed, on the other hand, that when the exciting circuit of an electromagnet is suddenly opened, the rise and decay of the Oeffnungsextrastrom induces in the mass of the iron, currents, alternating in direction and decreasing in intensity, and that the magnetization of a rod due to the original current is reversed in sign, under favorable circumstances, by a weaker current in the opposite direction. In the case of closed rings, where demagnetizing factors are absent, anomalous magnetization seems to appear only when eddy currents in the iron so shield the particles inside the mass that they are never exposed to sudden changes in the intensity of the exciting magnetic field.

My thanks are due to the Trustees of the Bache Fund of the National Academy of Sciences, who have lent me some of the apparatus used in measuring the oscillograms mentioned in this paper.

XX

THE ANOMALOUS MAGNETIZATION OF IRON AND STEEL[1]

In 1863, von Waltenhofen, who had been making experiments upon the retentiveness of bars of iron and steel for magnetism, discovered the phenomena which usually bear his name. If an increasing current (I), ending in the maximum value I', be sent through a long solenoid, the final value of the magnetic moment of a solid bar of originally demagnetized soft iron or steel within the solenoid frequently depends, not only upon the final strength of the current, but also upon the manner of growth of the current in attaining this intensity. The moment will be greater if the current be suddenly applied in full strength than if it be made to grow slowly, either continuously or by short steps. If, after the current has remained steady for a short time, at the strength I', it be made to decrease to zero, the residual moment of the bar will usually be less if the current be suddenly opened than if the decrease be made slowly, by gradually introducing more and more resistance, and the demagnetizing factor of a given cylinder is considerably less when computed from observations made by the method of sudden reversals, than if it is determined by slow, step-by-step changes in the exciting current.

Von Waltenhofen also encountered some cases of apparently anomalous magnetization which he describes in the following words taken from his paper in Volume 120 of Poggendorff's Annalen:

"Es ist mir oft aufgefallen, dass die magnetischen Rückstände in weichen Eisenkernen bei wiederholter, ganz gleicher, temporärer Magnetisirung desselben Stabes, sehr ungleich ausfallen. Noch befremdender aber war mir eine Erscheinung, die ich an einem sehr dicken Eisencylinder zuerst wahrgenommen habe, und welche darin bestand, dass der nach Aufhebung des magnetisirenden Stromes zurückgebliebene Magnetismus im Vergleiche mit dem verschwundenen tempo-

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlvii, no. 17, March, 1912.

rären Magnetismus manchmal sogar die entgegengesetzte Polarität hatte. . . . In der Magnetisirungsspirale wurde ein vollkommen unmagnetischer Cylinder von möglichst weichem Eisen, 103 mm. Länge und 28 mm. Durchmesser, mit zunehmender Stromintensität soweit magnetisirt, dass sein temporäres Moment nahe zu = 60 war. Nach plötzlicher Stromunterbrechung äusserte er das entgegengesetzte remanente Momente −0.20, und zeigte auch nach wiederholten plötzlichen Oeffnungen der wieder geschlossenen Kette entschieden negative (anomale) Rückstände. Dagegen zeigte sich nach allmählich eingeleiteter Aufhebung des magnetisirenden Stromes jedesmal ein bedeutendes, mit dem temporären Momente gleichnamiges Residuum. Wenn der Strom hierauf in derselben Richtung abermals hergestellt, sodann aber plötzlich unterbrochen wurde, zeigte sich das mit der temporären Magnetisirung gleichnamige Residuum, welches nach allmählicher Stromaufhebung immer wenigstens den Betrag 0.30 hatte, nahezu auf 0 reducirt; konnte jedoch durch Wiederholung dieses Verfahrens nicht merklich unter 0 herabgebracht werden. Wenn aber hierauf die magnetisirende Stromrichtung gewechselt wurde, so trat nach plötzlicher Unterbrechung wieder eine ganz entschiedene anomale Magnetisirung auf. So oft der Eisencylinder mehrere Tage in ostwestlicher horizontaler Lage unberührt gelassen war, zeigte er sich wieder vollkommen unmagnetisch, und ergab bei Wiederholung des zuerst beschriebenen Versuches wieder das anomale Residuum −0.20. Wenn dagegen die remanenten Magnetismen nicht durch längeres Liegenlassen, sondern durch entmagnetisirende Ströme verschwinden gemacht worden, gelang es nicht, so auffallende anomale Magnetisirungen hervorzubringen."

Seventeen years after the publication of von Waltenhofen's results,[1] Righi,[2] who was apparently unacquainted with his work, printed in the *Comptes Rendus* an account of some similar experiments of his own. He says: "On sait que le rapport entre le magnétisme rémanent et le magnétisme temporaire d'une barre d'acier enveloppée par une bobine magnétisante devient de plus en plus petit si la barre est de plus en plus courte et grosse." "Si l'on prend des barres d'un même acier et de même diamètre mais de longueurs décroissantes, on doit

[1] A. von Waltenhofen, *Pogg. Ann.*, vol. cxx, 1863.
[2] A. Righi, *Comptes Rendus*, vol. xc, 1880.

arriver à une certaine longueur qui ne donne pas de magnétization, pendant qu'avec des longueurs moindres on doit obtenir une polarité rémanente opposée à celle de la bobine." "Si le courant est très fort, le phénomène de la polarité anomale ne se produit qu'après avoir magnétisé la barre quelquefois dans les deux sens."

During the last thirty years, many persons [1] have studied the von Waltenhofen phenomena as they affect the hysteresis cycles of straight rods and of closed cores of iron, and some of these have discussed the bearing of their own measurements upon the theory of anomalous magnetization. G. Wiedemann [2] always maintained that his own researches and those of Righi showed that eddy currents in the iron, accompanying surges in the exciting circuit, accounted best for the observed facts. It seemed to von Waltenhofen, however, that the strange reversals of polarity which he had noticed could not be due to induced currents caused by sudden changes in the exciting circuit, and he explained them as consequences of the inertia of the molecular magnets turning in a viscous medium. Fromme,[3] Auerbach, Ewing, Peuckert, Zielinski, and others who have written upon the subject, seem to agree on the whole with von Waltenhofen's views.

The present paper attempts to throw some light upon the theory of the von Waltenhofen effect by a discussion of a number of experiments made for the purpose of determining the conditions under

[1] C. Fromme, *Wied. Ann.*, vol. v, 1878; vol. xiii, 1881; vol. xviii, 1883; vol. xxxiii, 1888; vol. xliv, 1891. F. Auerbach, *Wied. Ann.*, vol. xiv, 1881; vol. xvi, 1882; Winkelmann's *Handbuch der Physik*, Bd. v (214). W. Peuckert, *Wied. Ann.*, vol. xxxii, 1887. P. Bachmetjeff, *Rep. der Physik*, vol. xxvii, 1891. Zielinski, *Mitt. a. d. Telegraph-Ing.-Bureau d. Reichspostamtes*, vol. ii, 1896. Ruecker, *Inaugural Dissertation*, Halle-Wittenberg, 1905. Peirce, *Proceedings of the American Academy*, vol. xliii, 1907; vol. xlvi, 1911. L. A. Babbitt, *Ibid.*, vol. xlvii, 1911.

[2] "Schon bei Gelegenheit der von Righi wiederholten Versuche von v. Waltenhofen über die anomale Magnetisirung, hatte Ref. [Wiedemann] erwähnt, dass sich dieselben völlig aus dem Auftreten alternirender Inductionsströme in der Masse des Eisens beim schnellen Oeffnen des magnetisirenden Stromes u. s. f. ableiten lassen, von denen ein später auftretender weniger dichter, die Magnetisirung durch einen vorhergehenden dichteren Strom vernichten resp. umkehren kann. Die anomale Magnetisirung ist also *rein secundär*." — *Beiblätter der Annalen der Physik*, vol. v, 1881.

[3] "Gegenüber den Bemerkungen des Herrn Ref. [Wiedemann] halte ich meine Ansicht aufrecht, dass ich durch meine Versuche mit Eisendrahtbündeln, welche ebenfalls den Unterschied der permanenten Momente in der regelmässigsten Weise zeigten, schon nachgewiesen zu haben glaube, dass Inductionsströme keinesfalls zur Erklärung ausreichen können." — *Wiedemann's Annalen*, vol. xiii, 1881.

which anomalous magnetization appears. It leaves to a future article a consideration of some of the theoretical aspects of the subject.

The Demagnetizing of Stout Pieces of Iron or Steel

It is to be said at the outset that almost every piece of iron to be obtained nowadays in the market is more or less strongly magnetized when it comes into the hands of the observer, and that it is often very difficult, if not impossible, to demagnetize a massive block thoroughly. If a slender rod be placed inside a long solenoid in circuit with the secondary coil of a suitable open-core transformer, and if this coil be slowly drawn off the core with the help of some mechanical device, it is possible to send through the solenoid a long series of currents, alternating in direction and gradually decreasing in intensity, and thus to demagnetize the rod well enough for most purposes. The Jefferson Laboratory has three large sets of apparatus of this sort.

The process just described, however, does not succeed very well with stouter rods, for several seconds may be required to establish a steady current in the solenoid under a steady electromotive force if the core be large, and the use of alternating currents of commercial frequencies is barred out. The solenoid current may be reversed in such a case, at sufficiently long intervals, by means of a mercury commutator geared to an electric motor. Such a commutator, made several years ago by Mr. George W. Thompson, the mechanician of the Jefferson Laboratory, enabled Mr. L. A. Babbitt [1] to demagnetize very completely the finely divided core of a large toroidal transformer, though a number of hours were spent each time in the process. With irregular masses of metal this process also is often ineffective, and it is not always successful with short cylinders. A piece of soft Bessemer steel 5 centimeters long, recently cut from a long rod 3 centimeters in diameter, in the Jefferson Laboratory, was found to be slightly magnetized, and Mr. Thompson and Mr. John Coulson attempted to demagnetize it in a solenoid about 38 centimeters long, consisting of about 1460 turns of large wire. They began the series of alternately directed and slowly decreasing currents with one of more than 40 amperes, corresponding to a field within the solenoid before the iron was intro-

[1] *Proceedings of the American Academy*, vol. xlvii, 1911.

duced of about 1700 gausses, but the iron was still magnetized in the old direction, with nearly the same intensity, at the end of their work.

In demagnetizing a stout piece of iron by currents alternating in direction, it is well to put the metal slowly through a succession of complete hysteresis cycles with gradually decreasing ranges, but if this be inconvenient, the iron may be surrounded by a thick copper shell,[1] the eddy currents in which will prevent the magnetic changes in the

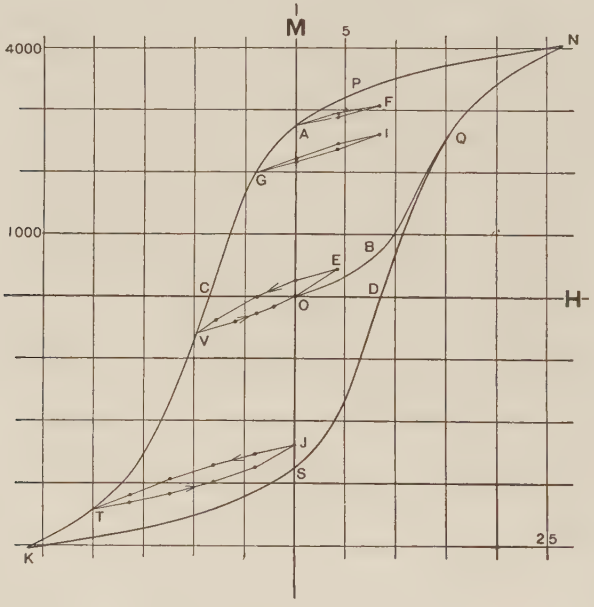


FIGURE 1

iron caused by a rudden reversal of the main switch from being so violent as they otherwise might be. As will appear more clearly in the sequel, the distribution of the magnetization in a stout iron cylinder in a solenoid which carries a current of given strength is different according as the current attained its final value slowly or suddenly, and it very much facilitates the demagnetization of such a piece, if the currents be applied slowly and decreased gradually.

It is often assumed that a piece of iron may always be completely demagnetized by heating it uniformly nearly to a white heat, main-

[1] Shuddemagen, *Proceedings of the American Academy*, vol. xliii, 1907.

taining it at this high temperature for some time, and then allowing it to cool slowly in a place where it will not be exposed to any magnetic forces; but in practice the procedure often fails, especially with material which has once been irregularly magnetized or which is not quite homogeneous. The spherical shield of a new Du Bois-Rubens Panzer Galvanometer in the Jefferson Laboratory proved to be slightly magnetized and consequently useless for the purpose for which it was made. This was heated to nearly a white heat, kept hot for about half an hour, and then very slowly cooled in a protected space, without causing it to lose appreciably its original magnetization; a repetition of the process was also unsuccessful.

Most of the specimens mentioned in the experiments discussed below were packed, one or two at a time, in fine iron filings, enclosed in a piece of large iron pipe provided with screw caps at the ends, and then heated thoroughly for some time, under a power blast, in a gas furnace. The pipe was surrounded by fire bricks and after the fire had been removed it was allowed to cool for many hours with its axis perpendicular to the earth's meridian before the annealing process was regarded as complete. In this manner most of the pieces were fairly well demagnetized. Of course, the permanent magnetic moment of an iron cylinder of length only twice as great as its diameter, is never very strong, but it was usually possible to detect some evidences of magnetization in every piece tested. A stout cylinder acquires a fairly large temporary moment, even when it is held with its long axis perpendicular to the earth's field, and it is very necessary to adjust the relative positions of such a specimen and a magnetometer by which it is to be tried, so that this magnetization shall not affect the measurements. The short iron cylinders which von Waltenhofen used must have been very soft indeed if they really lost their magnetization completely when left to themselves, with their axes perpendicular to the meridian, for a number of days.

Cases of Magnetization which are not Really Anomalous

In many cases of so-called "anomalous magnetization," it is evident that a strong magnetizing field applied in one direction has been succeeded by a weaker field in the opposite direction, and when this latter has been removed, the magnet has the polarity of the first field. This

is of course not wonderful except as we may regard all hysteresis phenomena as mysterious. Figure 1 shows a hysteresis diagram for an iron rod about 80 diameters long, with a number of loops corresponding to "side trips" within the main figure. It is clear that if the rod had been magnetized by a positive current so that the mag-

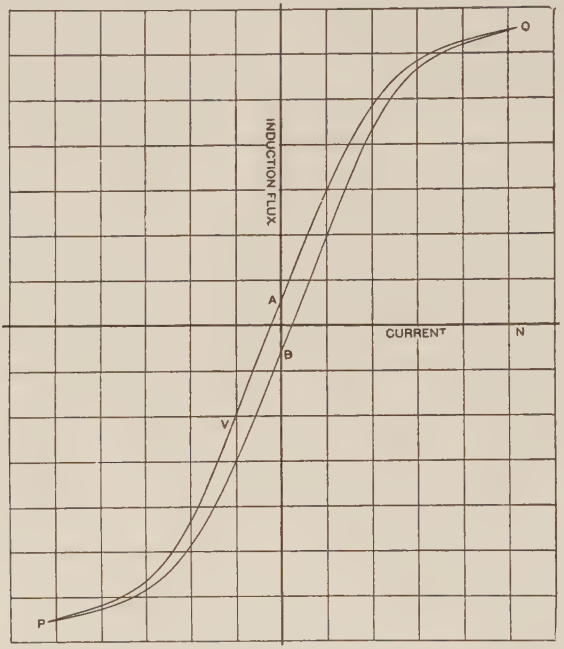


FIGURE 2. — Magnetic cycle for a piece of Seebohm and Dickstahl's special magnet steel 12 centimeters long and 3.5 square centimeters in cross section. If the exciting current be dropped from its highest strength to 0, and then reversed and given such a strength as to carry the magnetism to the negative value represented by the point V, the residual magnetism will be slightly positive when the current is taken off.

netic condition while the current is flowing is represented by the point N, and if the current be then stopped so that the condition of the rod is denoted by A and an oppositely directed current which gives rise to a field not greater than the abscissa of the point V be applied and then removed, the resulting magnetization of the rod will be represented by some point of the line OA. If the negative field is greater than OC,

the polarity of the rod while the current is on will be negative, but if it be not too strong the polarity will be positive when the field is off. This phenomenon is relatively pronounced in the case of a short, stout rod where OA is short and the slope of the lower side of an inner loop is almost parallel to the line KVAN which may be nearly straight. An example will make this statement clearer.

Table I gives the material for a kind of hysteresis diagram for a certain round rod of hardened tool steel, 2.8 cm. in diameter, and 12 cm.

TABLE I

H	N	H	N
960	551	0	+ 24
900	530	− 60	− 18
840	509	−120	− 61
780	489	−180	−103
720	467	−240	−137
660	443	−300	−191
600	420	−360	−231
540	392	−420	−272
480	361	−480	−316
420	321	−540	−354
360	279	−600	−391
300	237	−660	−422
240	195	−720	−451
180	144	−840	−504
120	110	−900	−528
60	66	−960	−551

long, when magnetized in a certain long solenoid. The column headed "H" gives the strengths in gausses which the fields within the solenoid would have if the rod were not there; the strengths of the exciting fields in the rod would be very hard to determine even if they did not vary from point to point in the metal. The numbers under "N" are proportional to the flux strengths through the central cross section of the rod.

If a positive current corresponding to $H = 976$ were sent through the solenoid and were then stopped, a negative current corresponding to $H = -35$ would make the magnetization of the rod negative while running, but a negative current corresponding to $H = -150$ could be used without making the magnetization negative when the current was taken off. A case like this may puzzle the observer if he does not happen to know that the rod he is using — which does not seem to be very strongly magnetized — has in fact been exposed

to some very intense field in the process of manufacture, but this phenomenon is very different from the one which von Waltenhofen describes.

The Anomalous Magnetization of Short Cylinders

The most characteristic examples of really reversed magnetization are to be found, perhaps, among short, stout rods of soft iron and steel, as von Waltenhofen and Righi explained many years ago. If such a rod, originally annealed and demagnetized, be placed within a long solenoid and be subjected to a magnetizing field of suitable strength, and if the exciting current be then gradually reduced to zero by the introduction of more and more resistance into the circuit — by very small steps, if not continuously — the remanent magnetism will have the same sign as, but only a small fraction of the strength of, the magnetization induced in the rod when the current was running. If, however, the current be suddenly interrupted by the opening of the circuit, it frequently happens that the sign of the residual moment is opposite to that while exposed to the field. Figure 3 shows a typical case of a certain kind, that of a solid piece of carefully annealed "Cold Rolled Shafting" 8 centimeters long and 3 centimeters in diameter. The "demagnetizing effect of the ends" in a rod of these dimensions is, of course, very great, and the residual moment in this instance was for many currents less than one per cent of the moment originally induced in the metal. The long, uniformly wound solenoid used for this experiment had a number (n) of turns per centimeter of its length such as to make $4\pi n/10$ almost exactly 25. In the figure, which represents a large number of observations, the horizontal unit corresponds to a field of 100 gausses if the iron were taken out of the solenoid. The actual field to which any portion of the rod was exposed for any value of the exciting current was of course difficult to determine. When the exciting current was suddenly broken a small spark appeared, and if the strength of the current was not greater than about 30 amperes, the remanent moment was of reversed sign.

The rod was at the outset in a nearly neutral condition. A small current (i) was at first applied to the solenoid, and then, with the help of a set of high-resistance rheostats made by the Simplex Electric Company, the rod was put many times through a hysteresis cycle with

this current, positively or negatively directed, to mark the limits. After this the same current was slowly applied again and gradually removed, and the remanent induction through the central cross section of the rod was measured by means of a small test coil of very fine insulated copper wire, so mounted that it could be quickly slipped off the rod and removed from the solenoid while the rod itself remained

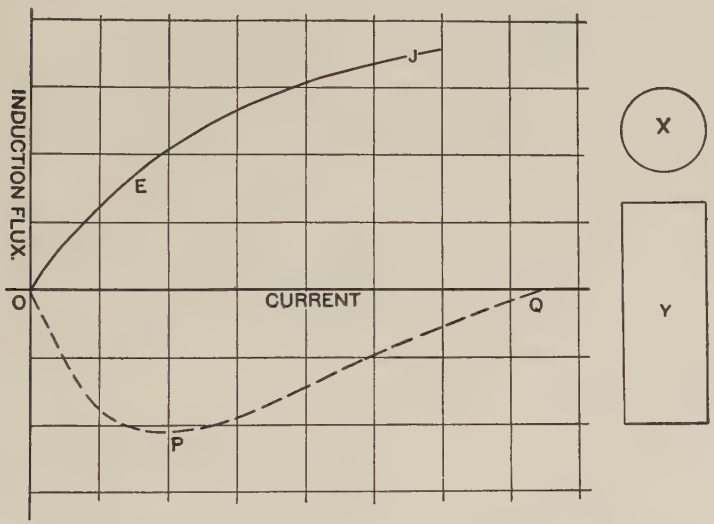


FIGURE 3. — The ordinates of these curves show the magnetic condition of a soft cylinder of mild steel 8 centimeters long and 3 centimeters in diameter, when the exciting current had been taken off — gradually, in the case of the full curve — suddenly in the case of the curve OPQ in which the negative ordinates indicate anomalous magnetization.

in situ. This flux was found for both directions of the magnetizing current. Then the current was applied gradually as before and the circuit was broken by a sudden blow upon a simple switch, and the remanent flux was determined for both directions of the exciting field. The ballistic galvanometer used was a low-resistance d'Arsonval instrument made for the purpose by Mr. Coulson, who worked with me in making all the observations recorded here. The scale distance was about five meters, and the telescope was focussed upon a real image of the scale formed about two meters in front of the object glass, by a lens used as the cover glass of the galvanometer mirror.

The process just described was repeated for each of a series of currents of increasing intensities, and in this manner material for the curves shown in Figure 3 was obtained.

It is, of course, possible to use a magnetometric method in testing the residual magnetism of short rods, but although we have made a large number of determinations in this way, we have found it inconvenient for several reasons. It will appear later that the lines of magnetization in the case of an anomalously magnetized rod are so folded together that the external effect of the remanent magnetism is usually small, and if a conveniently large magnetometer deflection is to be obtained, the rod must be very near to the needle. It is not safe to remove the iron from the solenoid in order to test it outside, for a slight blow might seriously alter the moment, and if the magnetism is to be measured while the specimen is in its place within the solenoid, the magnetometer must be set up near one end of the solenoid, where it will be violently disturbed by the exciting currents and the fields incident to the process of forcing the iron so many times through the hysteresis cycles by which it is prepared for the tests. If a stout specimen of soft iron is placed with its axis horizontal and perpendicular to the meridian, a moment large compared with the residual moment to be measured is induced in it by the earth's field, and it is practically difficult to prevent this transverse magnetization from masking the effect to be measured.

In all the observations mentioned in this paper the solenoid was placed with its axis perpendicular to the meridian, and in all but a very few instances to be mentioned specially, the rod was tested while inside the solenoid.

In almost every instance, also, the exciting current was applied slowly to the magnetizing coil. That is, the circuit was closed with a very much greater resistance in it than was finally needed, and this was gradually reduced to the proper amount. If the circuit was suddenly closed with this final resistance in it, the residual moment of the rod was much smaller in absolute value, whether the current had been gradually or suddenly reduced to zero, than if the rise of the current had been slower.

Figure 4 shows the results of tests similar to those described above, but made upon a phenomenally soft bar of mild steel 12 centimeters

long and about 2.86 centimeters in diameter. The line OEC, which is nearly straight, represents the induction flux through the central cross section of the bar while the metal was under the action of the magnetizing field. Each ordinate has only one nine-hundredth of the

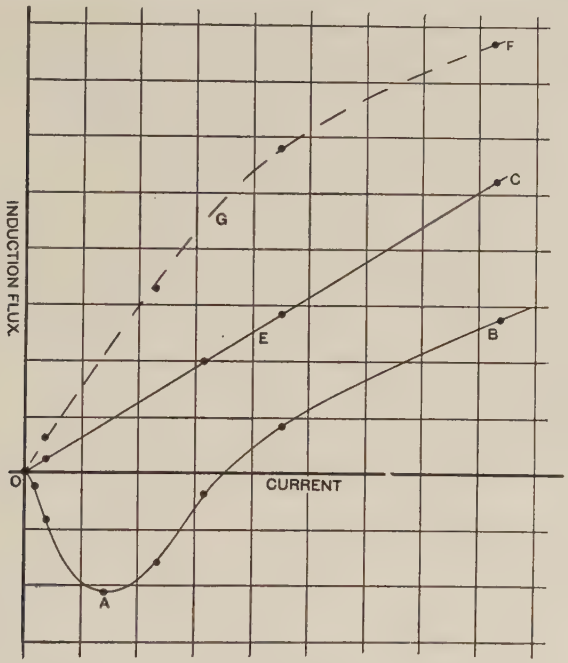


FIGURE 4. — OGF and OAB show the flux of magnetic induction through a rod of very soft steel 2.86 cm. in diameter and 12 cm. long, after the magnetizing field had been gradually and quickly removed, respectively. OEC shows on a reduced scale the flux in the rod while it was exposed to the magnetizing field.

length it would have if the scale of this curve were the same as for the other lines in the figure. The ordinates of OGF give the remanent flux when the slowly applied exciting current was as slowly reduced to zero. The line OAB shows the residual flux after the current had been suddenly interrupted. The solenoid used in this work has 1460 turns in a length of 47 centimeters. The horizontal unit in the diagrams corresponds to about 80 gausses for the field ($4\pi n\, I/10$) in the solenoid due to the current in its coil. A discharge of 1 microcoulomb

sent through the low-resistance galvanometer would cause a throw of 186 millimeters of the scale, and a throw of 1 millimeter corresponded to a flux of about 0.74 maxwells through the steel. The vertical unit in the diagram for the lines OGF, OAB, is 15 maxwells. It is evi-

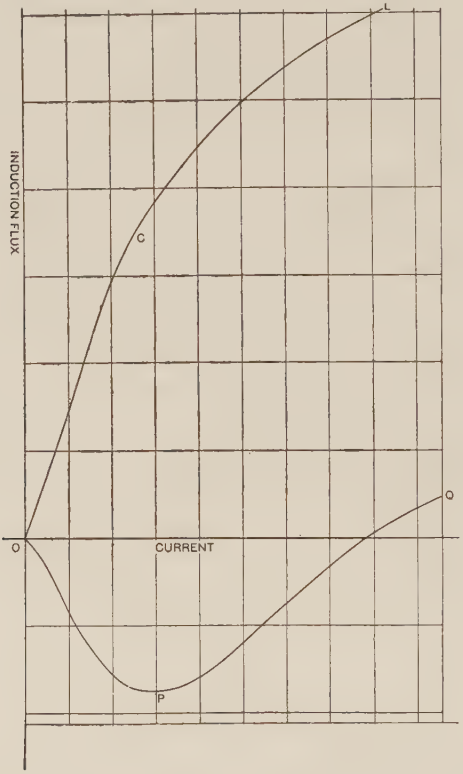


FIGURE 5.—A cylindrical shell, about 12 centimeters long and 2.8 centimeters in outside diameter, was exposed in a long solenoid to exciting currents of various intensities. OCL represents the flux through the central cross section of the cylinder when the current was running, and OPQ, drawn on a somewhat exaggerated scale, the remanent flux when the circuit was suddenly broken. The horizontal unit corresponds to a field of 25 gausses in the solenoid when the iron was not there.

dence of the extraordinary magnetic softness of this specimen that whereas the flux through the rod corresponding to the point C was 70,000 maxwells, this sank to 113 maxwells when the current was slowly removed.

Figure 5 represents the results of experiments upon a soft steel shell about 12 centimeters long, 2.83 centimeters in outside diameter, and 1.9 centimeters in diameter inside. OCL represents the flux through the central cross section of the shell when the current was running, and

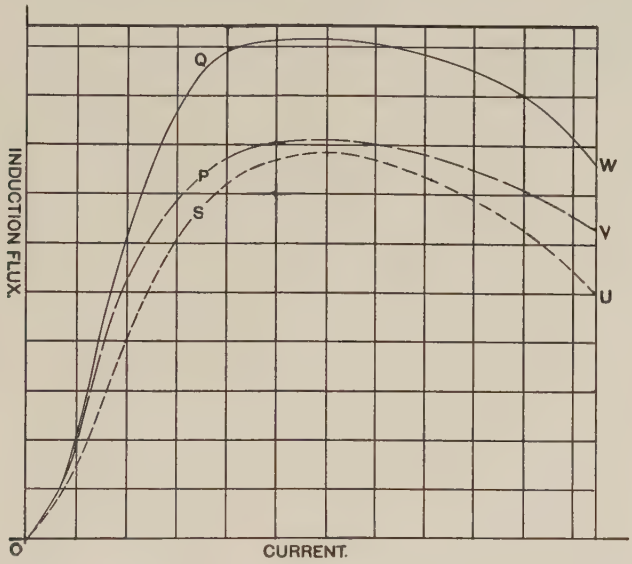


FIGURE 6. — The ordinates of these curves represent the residual magnetism in a certain short, stout piece of soft steel magnetized in a solenoid, when the exciting current had been suddenly interrupted. A positive ordinate indicates reversed or anomalous magnetization. The observations recorded in each curve were taken after the specimen had been newly annealed, and the differences between the curves show that it is difficult to demagnetize a cylinder of such dimensions completely.

the lower curve shows the residual flux upon a relatively larger scale. This residual flux is reversed for solenoid fields less than about 200 gausses.

The three pieces of steel to which the curves of Figures 3, 4, and 5 belong were all freshly annealed just before they were magnetized, and the same precaution was taken in the case of almost every other specimen mentioned in this paper. The most careful reannealing does not generally bring a stout piece of iron which has been exposed to a strong field exactly back to its original magnetic state, though the

differences are often so small as only to be discoverable when the specimen is tested for anomalous magnetization. Figure 6 shows such a test made upon a soft piece of Bessemer steel freshly annealed before the observations recorded in each curve. The residual moments were themselves very small and the differences were in absolute value very small indeed but are evidently real.

Figures 7 and 8 show the results of experiments made upon two pieces 12 and 8 centimeters long, respectively, cut from a rod of cold-rolled shafting about 3 centimeters in diameter. Each piece was exposed to a long series of magnetic fields alternating in direction and

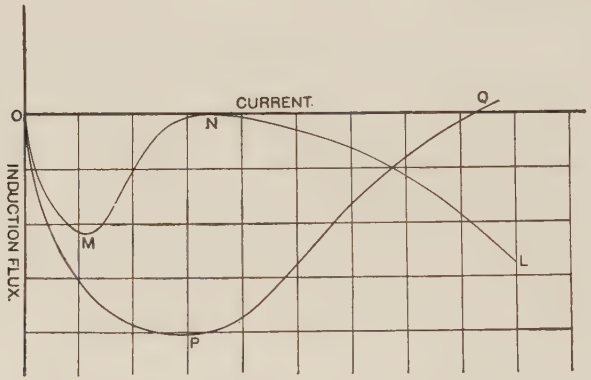


FIGURE 7. — This diagram shows magnetic bias in a short rod of cold-drawn mild steel. This resisted the ordinary processes for demagnetizing the iron.

gradually decreasing in intensity, with the hope that this process would remove any magnetization that the rod might have acquired in the making, but both pieces show a decided bias which was too strong to yield to such treatment. In Figure 7, OPQ is the residual magnetism after currents which have caused positive moments have been suddenly destroyed. OMNL shows the remanent magnetism after currents which have caused negative moments. The first curve indicates that the residual magnetism was reversed in sign, but this was never the case after negative currents. In Figure 8 similar curves are shown for the shorter specimen and it appears that some of the negative currents left anomalous residuals. Annealing removed the bias from the first piece almost completely.

Apart from details the observations recorded in this section are in general agreement with much that has been written upon this subject as given [1] in Wiedemann's *Elektricität*. Wiedemann denotes by T the total magnetic moment of a bar when exposed to the action of a magnetizing field and by P the residual moment after the field has disappeared. He uses the suffixes a and f to denote that the moment of which he is speaking has been reached by a gradual change in the exciting field or by a very sudden one. He says that T_f is always larger than T_a, and P_a than P_f, algebraically considered, but these differences are only large in short rods. P_f is slightly increased if the

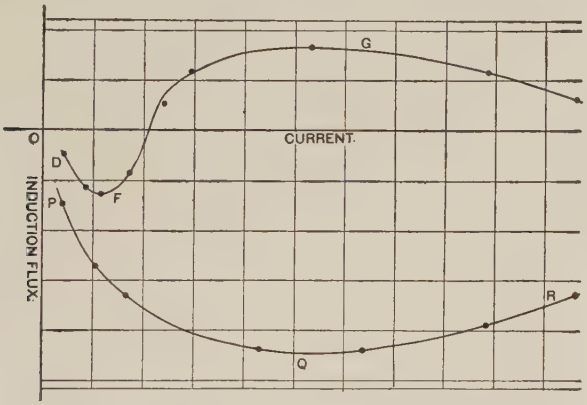


FIGURE 8. — Magnetic bias in a rod of cold-rolled shafting which the demagnetizing process used could not remove.

rod to be magnetized is surrounded by a thick tube of non-magnetic metal. $(P_a - P_f) / P_a$ is smaller for bundles of insulated soft iron wire than for solid rods of the same dimensions. Inversion comes with longer rods when the exciting field is weak than when it is strong. Some of these statements will need to be discussed in the light of experiments upon divided cores. The statement copied by Wiedemann that if the iron core has been already magnetized in the normal direction by a current which has been slowly brought to zero, anomalous magnetization does not occur when a second current is applied in the same direction and then suddenly stopped, runs counter to all our experiences. Specimens which have not been really demagnetized

[1] Bd. iv, §§ 338–340.

show, of course, all sorts of abnormal behavior, but we have found it easy, with suitable cores, to get reversals after a slowly applied current has been slowly removed, by breaking suddenly a current in the same direction whether this last was applied slowly or suddenly. If a quickly applied current has been slowly cut off, we can get reversals by quickly breaking a current in the old direction, applied either quickly or slowly. If we apply either slowly or quickly a current in a fixed direction, then open it suddenly, and repeat this process a score of times, the reversal usually occurs at every break of the circuit without any reversal in direction of the exciting current. The remanent magnetism after a slow break, is greater if the current was quickly applied, but, as we have seen and as Wiedemann's statements would lead us to expect, anomalous magnetism occurs more regularly if the current has been slowly applied.

As Righi pointed out in 1880, if one cuts a number of pieces of different lengths from a stout steel rod and, beginning with the longest and taking them in order, tests the sign of the remanent magnetization of each after the exciting field has been suddenly destroyed, one often arrives at a length where anomalous reversals begin and continue for shorter pieces. Figures 9, 10, 11, are founded upon a set of such tests made upon rods cut from the very soft bar which furnished the specimen to which Figure 4 belongs. The diameter of the bar was about 2.83 centimeters, and the lengths, in centimeters, of the pieces used were 40.1, 31.8, 20.9, 18.0, 13.6, 12.0, 10.0, and 8.0. Figure 9 shows the residual fluxes through the centers of the pieces for all the specimens except the first and the fifth. These are all reversed in sign, but the amounts are extremely small because of the remarkable softness of the material. The horizontal unit corresponds to a solenoid field of 20 gausses, the vertical unit is about 7 maxwells. The first piece, 40 centimeters long, showed a slight reversed moment for excitations in the solenoid up to about 38 gausses, but the ordinates of the positive loop were not so high as the other curves of the series might lead one to expect them to be. Figure 10 shows the residual fluxes after the exciting currents had been slowly reduced to zero. The horizontal unit is here 80 gausses and the vertical unit 30 maxwells. Figure 11 shows the induction fluxes through the cross sections of the rods while they were in the magnetizing fields. The horizontal unit

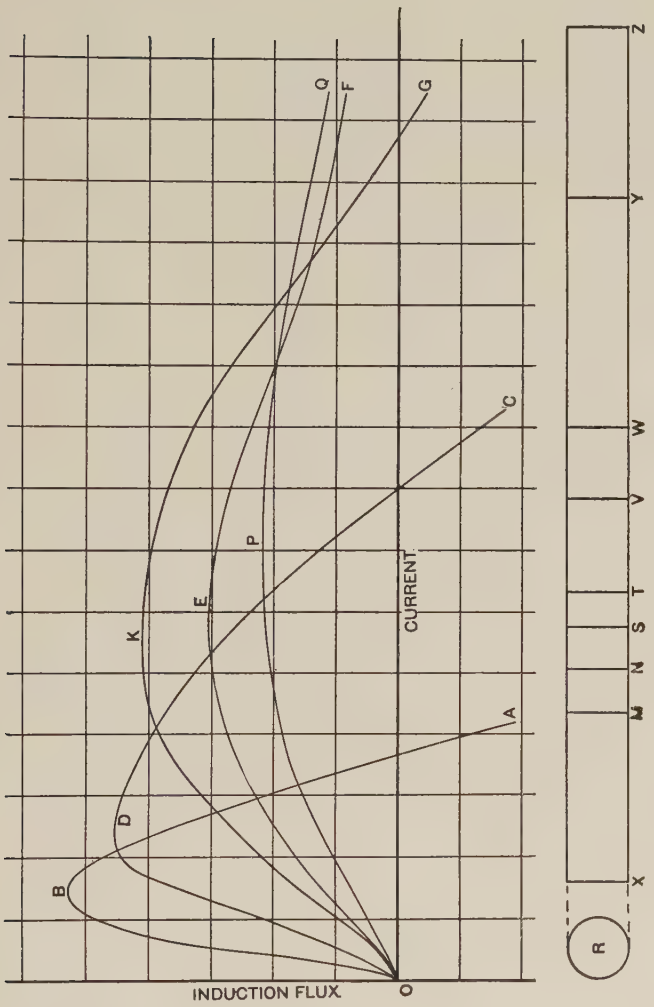


FIGURE 9. — Abnormal residual magnetization in rods of very soft mild steel, about 3 centimeters in diameter and of different lengths.

is in this case 80 gausses and the vertical unit very nearly 14,000 maxwells. Of 42,000 maxwells which the 8 centimeter long piece had under an exciting current of 16.4 amperes, only about 55 maxwells remained when the current had been gradually destroyed, and this for a cross section of about 7 square centimeters.

It is usually difficult to get a piece of Bessemer steel 3 centimeters in diameter and even 30 centimeters long so soft, magnetically consid-

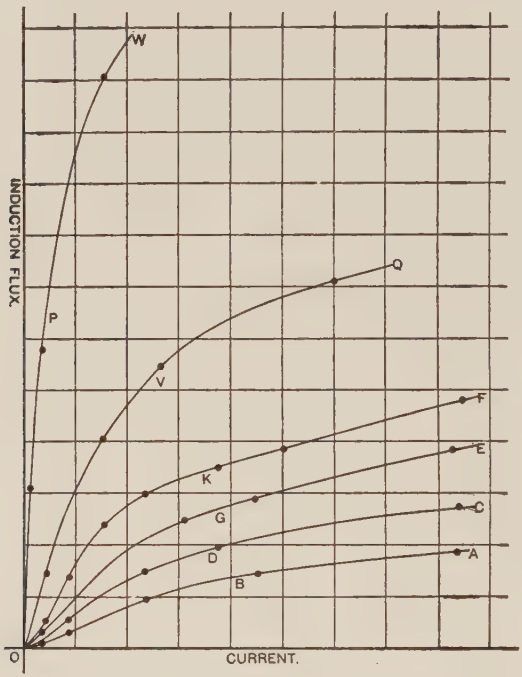


FIGURE 10. — Residual magnetism of normal sign in rods of very soft mild steel of different lengths.

ered, that it will show abnormal residual magnetism, and the metal just mentioned is exceptional, as has already been said.

Pieces cut from a certain round rod of soft steel, of lengths in centimeters 30, 15, 8, 6.8, and of diameter 1.6 centimeters, all refused to reverse when tested, but reversal finally appeared in a piece about 6.4 centimeters long.

It is often impossible to make a stout piece of hardened tool steel

reverse unless its length be made so small that the observations become doubtful. The next table (Table II) records the results of observations made upon a certain piece of glass-hard tool steel, 12 centimeters long and 3 centimeters in diameter. The solenoid used in this experiment, one of a large number at our disposal, was 176.2 centimeters long and had 5526 turns of insulated wire divided up into three coils of 1837, 1847, and 1847 turns respectively. The first column

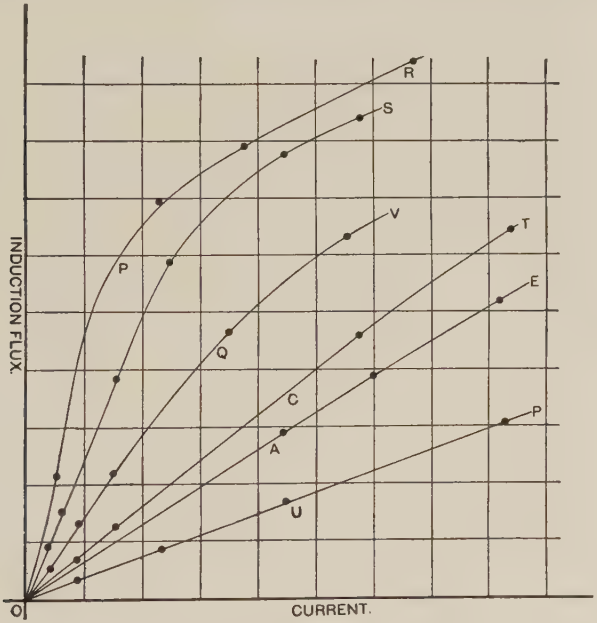


FIGURE 11. — Magnetism induced by different exciting fields in a set of rods of various lengths.

gives the intensity of the exciting current and the second column, on an arbitrary scale, the remanent magnetism, which always had the sign of the magnetizing field which had just been suddenly interrupted.

TABLE II

I	N		
0.2	2	2.0	98
0.4	6	2.5	148
0.6	17	3.0	196
0.8	23	4.0	302
1.0	31	5.0	415
1.5	61		

There is here no trace of anomalous magnetization.

If a strong current running through a solenoid containing a core of magnetizable metal, be suddenly broken, there will be, under favorable circumstances, an oscillatory discharge across the spark gap, and according to Wiedemann's theory, which was based upon the numerous experiments of early observers [1] upon the magnetization of steel needles by discharges from Leyden jars, the phenomena of anomalous magnetization are to be explained by the action of oscillating currents rapidly decreasing in intensity, induced in the outer portions of the core under test. Through the kindness of Mr. William Otis Sawtelle, who has a large revolving mirror driven by a powerful motor, and devices which he has himself designed and constructed for photographing electric sparks under various conditions, we were able to make sure that in the cases of the apparatus which we used in our experiments upon anomalous magnetization the discharge, when we suddenly opened the circuit, was uniformly oscillatory in character. Mr. Sawtelle and Mr. Coulson photographed a large number of these sparks; and, from their results, there cannot be any doubt, I think, that there were usually several hundred reversals in direction while the visible discharge lasted. With one of our solenoids, the period proved to be about 1/58000th of a second, and Professor G. W. Pierce, who most kindly tested one of our coils by itself, showed that such frequencies were to be expected. In each of the spark photographs, the record crossed the plate many times and the growth of the spark length with the time when the circuit was suddenly opened could be studied from them. It appeared that the manner of throwing the circuit open had very little effect upon the character of the discharge. When the current in the solenoid circuit is brought to zero by the continuous introduction of more resistance into the circuit, we do not expect that alternative currents will be induced in the core. It is difficult, however, to get any satisfactory theory upon which to base a mathematical investigation of the results of currents induced in a core of soft iron by oscillations decreasing in amplitude in a neighbor-

[1] Savary, *Ann. de Chim. et de Physique*, vol. xxxiv, 1826. Von Liphart, *Pogg. Ann.*, col. cxvi, 1862. Paalzow, *Pogg. Ann.*, vol. cxvii, 1862. Reiss, *Pogg. Ann.*, vol. cxxii, 1864. J. Henry, *Scientific Writings*, pp. 203 and 293. Rayleigh, *Phil. Mag.*, vols. xxxviii, xxxix, 1870. Rutherford, *Phil. Trans.*, vol. clxxxix, 1897. Wilson, *Electrician*, vol. li, 1903. Fleming, *Proc. Roy Soc.*, vol. lxxiv, 1903.

ing circuit. Even if the courses of such currents in a non-magnetizable core could be satisfactorily treated, and this seems difficult without a more accurate knowledge than we have about the behavior of the exciting current oscillations, we should not have any clear light upon what happens in a core magnetized in lines which are often closed within the metal, after the magnetizing current has been removed and the changes which come from the rapidly changing demagnetizing forces from the ends of the core itself are going on. We may content ourselves at present, therefore, by showing that, so far as we know, oscillations are always present in the circuit of the exciting current when anomalous magnetization is afterwards to be detected in the core. We must not close our eyes, however, to the fact that the demagnetizing forces due to the magnetic distribution itself complicate the problem.

Residual Magnetization in Bundles of Fine Iron Wire

The remanent magnetism in a bundle of fine iron wire so shellacked as to prevent electric flow from one wire to the next, should be interesting because the effects of eddy currents in the core itself are nearly avoided. Fromme's work in this direction seems not to have been conclusive, and it will be instructive to consider two or three experiments.

Two similar solenoids were placed horizontal with their common axis perpendicular to the meridian, and with their nearer ends about 15 centimeters apart. These solenoids were so connected in series that a current sent through the circuit did not affect the needle of a magnetometer between them. A bundle of fine, varnished iron wire forming a cylinder 12 centimeters long and 3 centimeters in diameter was then introduced into one of the solenoids and tested to make sure that it had been properly demagnetized. A small current was next sent through the circuit and the wire put several times through the cycle corresponding to this current. Then the circuit was suddenly broken so as to bring the current from its full value to zero and the needle deflection caused by the residual magnetism was observed. This process was then repeated for a series of currents of increasing strengths. The results of the work are given in Table III. H represents the strength which the current would cause in the solenoid if the

disturbing effects of the iron itself were not present. D shows the deflections of the needle on its scale caused by the residual moments. It is evident that there was nothing here similar to the abnormal magnetization of a soft iron solid cylinder of the same dimensions under similar conditions.

TABLE III

H	D	H	D
10	19	150	182
19	45	211	208
37	81	301	233
63	112	420	253
89	138	530	263
110	156		

A bundle of the same size as the last (12 centimeters long and 3 centimeters in diameter) was then made of pieces 12 centimeters long cut from Bessemer steel wire some of it 2.4 millimeters in diameter and some of it twice as large. In this case the wires were not varnished and eddy currents were not wholly prevented. The observations were made by determining the induction flux through the central cross section of the bundle, first, when the exciting current was running, and then after it had been suddenly destroyed. The first column in Table IV gives the strength of the field in the solenoid ($4\pi nI/10$); the other columns give, on an arbitrary scale, the flux values.

TABLE IV

H	N	N′
10	293	6.8
19	592	15.0
36	1144	30.1
60	1900	52.0
84	2740	71.6
108	3490	87.2
145	4700	113
201	6900	140
290	9200	166
400	13800	188
501	17300	210

These results represent very fairly all our experiences with bundles of iron wire. Although most transformers show the von Waltenhofen phenomena unless the cores are very minutely divided, I have never been able to get even an approach to a reversal of sign of the magnet-

ism of the short packages of fine wire that I have used. One of these, which was about 3 centimeters in diameter, was only 6.8 centimeters long.

In a stout iron cylinder made up of a small number of large pieces, anomalous magnetism is frequently to be found. Figure 12 shows the results of an interesting test upon a short cylinder of soft Bessemer

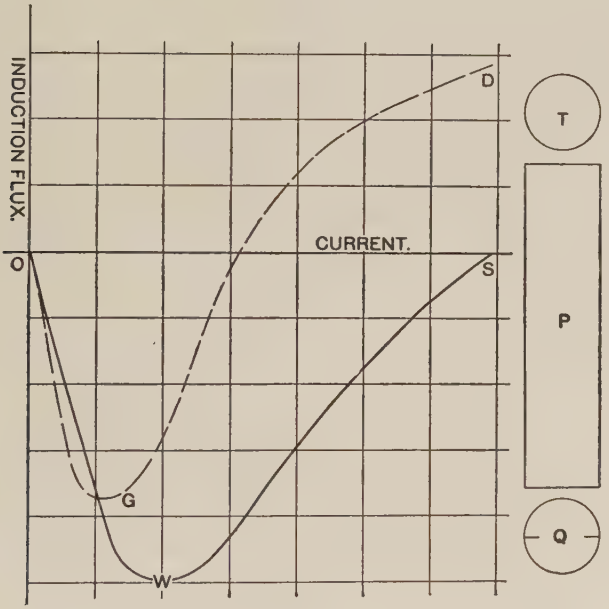


FIGURE 12. — This diagram shows anomalous residual magnetism in the case of a piece (P) of soft Bessemer steel, 12 cm. long and 2.8 cm. in diameter. The full curve was obtained with the cylinder intact as shown at T, the dotted curve, after the specimen had been slit lengthwise in the manner shown at Q.

steel, at first solid and then slit in a milling machine lengthwise with a very thin saw. The forms of the curves which show the magnitudes of the anomalous magnetization in these cases are similar, but the effect of the slits is very marked.

As has been already explained, we usually opened a circuit, when this had to be done suddenly, by a sharp blow upon a switch, but we experimented with other devices without finding that any of them

was better. At one time we broke the current by shattering a short piece of glass-hard steel wire introduced for the purpose into the circuit, but we did not discover that this process led to different conclusions from those which we reached with the more convenient key.

Anomalous Magnetization in Cylinders Formed of Shells and Cores

As will appear more clearly in the sequel, many of the lines of polarization in a short, anomalously magnetized solid cylinder form

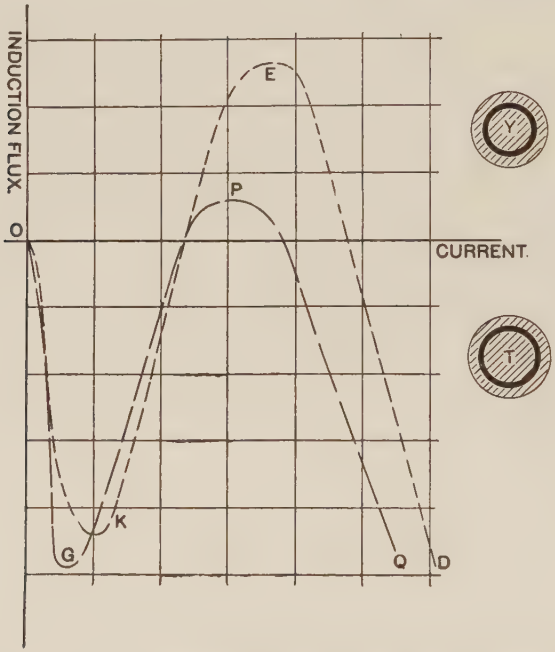


Figure 13. — The ordinates of each of the curves of Figure 13 show the remanent magnetism in a combination of a soft Bessemer shell and core when the slowly applied exciting current has been suddenly broken. A negative ordinate indicates that the residual magnetism has a sign opposite to that of the field in the solenoid when the current was running.

closed curves wholly inside the metal, and a cut made in the iron in the form of a cylindrical surface coaxial with the surface of the specimen would seriously interfere with this arrangement because there would be a very sensible reluctance at the crack. We should expect,

therefore, that the magnetic characteristics of an iron cylinder formed of a cylindrical core and a coaxial shell would be in some respects different from that of a solid cylinder, and this is the fact.

Figure 13 gives two curves, the first, OKED, belonging to a shell of diameters 2.83 and 1.93, with a core of diameter 1.60 centimeters; the second, OGPQ, to a shell of diameters 3.20 and 2.20, with a core of diameter 1.90. In each of these cases the residual magnetism is reversed in sign for comparatively small currents, then direct for cur-

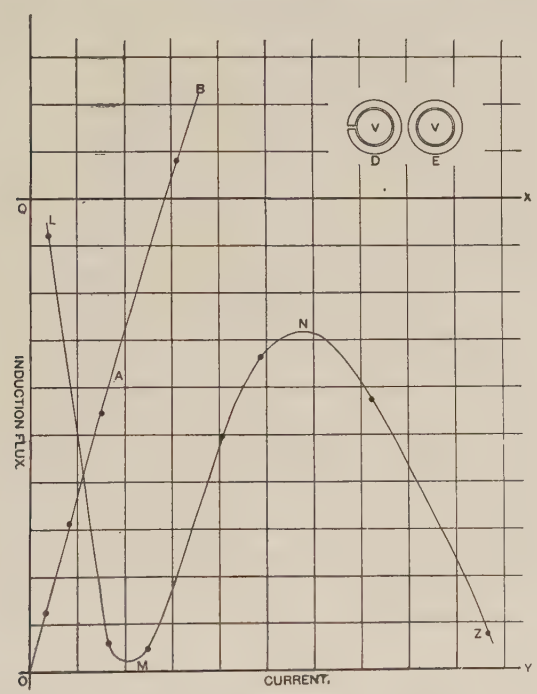


FIGURE 14. — Two similar shells of soft steel, one intact, the other slit lengthwise by a thin saw cut, were used successively over the same core. LMNZ and OAB show the fluxes through the combinations in the two cases.

rents somewhat stronger, and for large currents is again reversed with no apparent desire to become again positive. Curve LMNZ of Figure 14 belongs to a shell of diameters 2.83 and 1.93 with a core of diameter 1.90. The gap between core and shell is here narrower than in the cases just mentioned, and the curve which gives the magnitude

of the residual magnetization in terms of the exciting current, while of the same general form as those of Figure 13, does not cross the axis of abscissas, and the residual moment is reversed for all the excitations shown in the curve.

Figure 15 shows some observations made upon a shell and core which were not very successfully demagnetized. A slight bias exists:

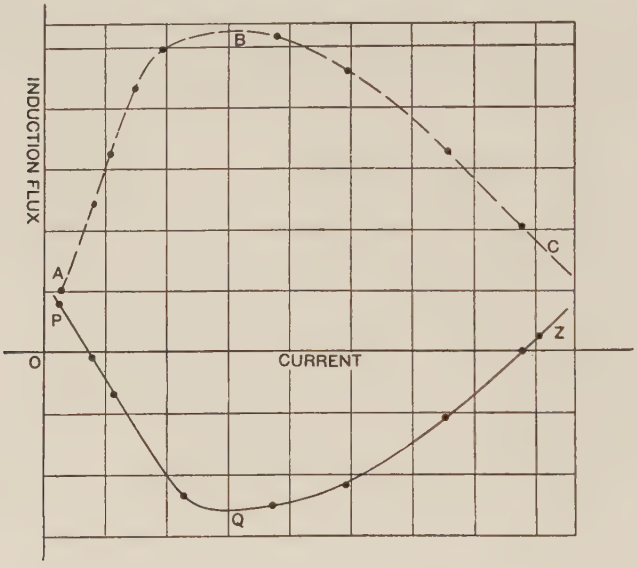


FIGURE 15. — A combination of shell and core, made of soft Bessemer steel, was demagnetized as completely as possible and then tested in a long solenoid. The magnetizing current was built up gradually and then suddenly broken. The ordinates of ABC show the residual flux through the metal when the current gave a negative moment before it was interrupted, the ordinates of PQZ show the remanent flux for oppositely directed currents. There was a bias in the specimen which showed that the process of demagnetizing it had not been wholly successful, and the anomalous magnetization is of different magnitude on the opposite sides.

ABC and PQZ show the residual fluxes through shell and core, the first for currents which give a negative moment while they are running, the second for positive currents.

Figure 16 shows how greatly the manner of building up the current, which is then to be quickly broken, affects the amount of the negative

or reversed magnetizations. Both of these curves show anomalous magnetization for moderate currents, but the residual flux is very much greater if the current is built up gradually than if it is built up suddenly.

The gap between core and shell in the combinations X and Y of Figure 13 and some others we have used, was purposely made wide enough to permit of the introduction of a very thin ring coil to embrace the core alone and thus make it possible to study separately the behavior of each part of the system. The results of experiments of this kind proved instructive, as will appear from an account of a typical case.

A cylindrical shell 12 centimeters long, the diameters of which were 3.00 and 2.27 centimeters, was used with a core 1.90 centimeters in diameter to form a combination (Z) which, after being thoroughly demagnetized, was placed in a long solenoid and exposed to a series of magnetizing fields, each a little stronger than the preceding. At every step, the metal was put a number of times through the hysteresis cycle corresponding to the exciting current employed, and then the fluxes through the central cross sections of core and shell were measured while the current was running. In Table V, H represents the field ($4\pi nI/10$) due to the current in the solenoid, F is the flux in maxwells through the combination of core and shell, N is the flux through the core alone, and N′ is the flux which the core would carry if the whole flux through the system were uniformly distributed. The area of the cross section of the core was about 47 per cent of that of the combination.

TABLE V

H	F	N	N′	N/F
10	1270	130	595	0.102
20	2500	215	1170	0.086
30	3670	295	1720	0.080
40	4830	365	2270	0.076
50	5990	430	2820	0.072
60	7150	500	3360	0.070
70	8310	565	3920	0.068
80	9470	630	4450	0.067
90	10640	700	5000	0.066
100	11820	765	5500	0.065
125	14800	930	6950	0.063

While the slowly built up current is running steadily the flux in the core, which should be nearly half that through the whole combination if the flux is to be uniformly distributed, is very much less.

When in the case of this combination (Z), the slowly built up current, so directed as to make the flux positive while it is running, is very

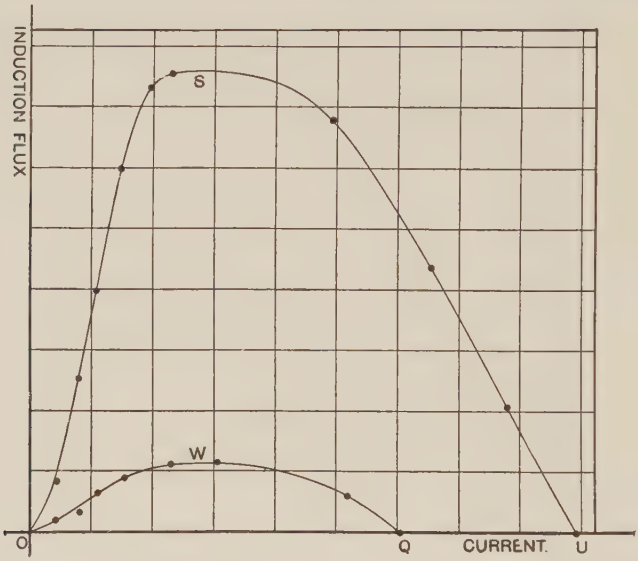


FIGURE 16.—A shell and core of soft Bessemer steel about 12 cm. long were exposed to a magnetizing field of given strength in a long solenoid, and the exciting current was then suddenly broken. The ordinates of the curves OSU and OWQ represent the remanent induction flux through the central cross section of the specimen, when the solenoid current was slowly built up and suddenly applied, respectively. A positive ordinate represents a reversal of magnetism in these curves.

slowly decreased to zero, the remanent flux through the system is positive, but the flux in the core becomes *negative*, in the manner indicated by the figures in Table VI in which F, N, S, are the induction fluxes through the whole combination, the core, and the shell, respectively. While the remanent flux through the system increases regularly with the strength of the current, the oppositely directed fluxes in the shell and the core decrease after reaching maximum values.

In all work with short bars of iron or steel, the "demagnetizing force due to the ends" becomes very important. The outer portions

of a very short magnet often reverse the direction of the polarization in the inner portion so that most of the lines of polarization are closed within the metal and the effect of the magnet upon a magnetometer needle is often very slight indeed. If a cylindrical shell with a core

TABLE VI

H	F	N	S
50	+ 86	−322	+408
100	+122	−410	+530
150	+132	−370	+500
200	+141	−318	+459
250	+149	−265	+415
300	+155	−213	+368

of loose steel wires be placed inside a solenoid of stout wire and a powerful current be then suddenly sent through the coil, the wires will be thrown violently out of the shell if the latter be short and stout, in a direction which shows that the moment induced in them by the exciting current was opposite in sign to that of the shell. This experiment is sometimes very striking.

If, in the experiments upon the combination Z, the exciting current was suddenly destroyed, the flux in the shell became instantly *negative*, while the remanent flux in the core was *positive* just as it was when the current was running through the solenoid circuit. See Table VII.

TABLE VII

H	F	N	S
10	− 7	+ 57	− 64
20	−18	+134	−152
40	−25	+180	−205
50	−21	+184	−205
100	− 5	+178	−183
150	+ 4	+170	−166
200	− 4	+171	−175
250	−17	+214	−231
300	−30	+320	−350

At $H = 1100$ gausses F gave a negative throw far off-scale, and N a similar positive throw. At this excitation, however, the solenoid current had to be so strong as to heat the coil rapidly and we did not attempt to make careful determinations of these fluxes. If F were plotted against H we should get a curve of the form shown in Figure 13.

All the combinations of shell and core that we have used give a set of fluxes for the F column which vary with the excitation in much the same way that the whole flux for Z does. There is always — so far as my knowledge goes — an increase in N from a low value near the outset to a rapidly increasing one at high excitations, but sometimes the increase is regular and sometimes not. As an instance of a very rapid increase in N beginning near a given excitation, I may cite the case of

TABLE VIII

I	F	N
0.22	− 1.8	+ 14
0.42	− 4.7	+ 33
0.82	−12.2	+ 78
1.44	−16.3	+118
1.77	−18.0	+138
2.25	−17.2	+148
2.73	−15.7	+162
3.43	−10.1	+177
5.00	+ 2.7	+197
7.60	+10.3	+308
12.10	−20.1	+664

a certain combination of nearly the same dimensions as Y of Figure 13, which was tested in a solenoid for which $4\pi n/10$ was very nearly equal to 25. The first column in Table VIII gives the intensities of the exciting currents used, the second and third columns the fluxes after the currents had been suddenly interrupted.

L in Figure 17 represents a combination of two coaxial shells and a core, very accurately made and carefully annealed by Mr. Thompson. The diameters, in centimeters, of the five cylindrical surfaces were

TABLE IX

I	Core	Inner Shell	Outer Shell
0.60	− 7.8	− 38	+ 58
1.41	− 25.6	−160	+225
4.70	− 56.5	−292	+428
8.50	− 84.0	−276	+474
17.00	−156.0	− 34	+294
44.50	− 41.4	+215	+ 47

1.12, 1.58, 2.53, 3.00, 3.96. This system was treated like all the other test pieces and the fluxes through all three members were determined after the currents which had been slowly applied had been slowly brought to zero and again after they had been suddenly destroyed.

Tables IX and X give the remanent fluxes for the slow breaks and for the quick breaks respectively. I represents the solenoid current in amperes.

TABLE X

I	Core	Inner Shell	Outer Shell
0.60	+ 6.2	+ 83	− 109
1.41	+ 12.1	+ 174	− 223
4.70	− 2.5	+ 299	− 318
8.50	− 0.7	+ 364	− 366
17.00	+ 1.9	+ 652	− 674
44.50	+155.0	+1445	−1922

The signs are nearly all different according as the current is slowly or quickly destroyed.

The observations already described represent fairly all our work upon combinations of solid shells and cores and it remains to mention

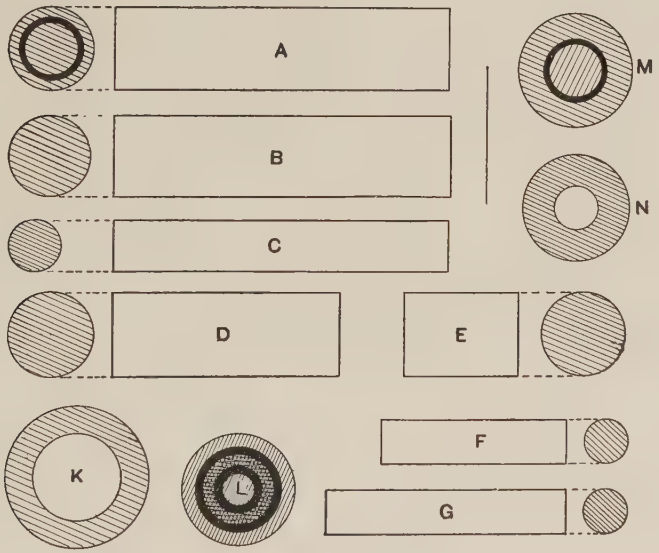


FIGURE 17. — The forms of some of the test pieces.

the special case represented by the nearly straight line of Figure 14. Here the shell was of the same dimensions and of similar material as that used in the work which led to the curve LMNZ in the same figure, but this shell was slit through lengthwise by a single saw cut which prevented currents from circulating around it. Many of our

specimens were made 12 centimeters long so that our observations might be more easily comparable with some which von Waltenhofen and Fromme made.

We have seen that if in a combination of a shell and a core the exciting current be gradually reduced to zero, the residual magnetization of the shell is usually normal and that of the core reversed. If, however, the current is suddenly broken, the magnetization of the shell is often reversed and that of the core is normal. These facts may be proved by removing the specimen from the solenoid and testing the two pieces which then seem to be strongly magnetized, separately with a compass or magnetometer. The separation of the members of the system alters the polarization in each, however, and the process is not to be recommended in accurate work.

It is very difficult to study the residual magnetism in a very short, stout soft iron cylinder, whether this be normal or anomalous, by the use of iron filings, for since so many lines of polarization are closed within the metal, the external action of the magnetization is usually small. In the case of a combination of a shell and a core, where the gap prevents the arrangements of polarization from being what they would be in a solid cylinder of the same dimensions, it is often practicable to show by the aid of very fine filings that lines emerge into the air from the outer filaments at one end of the specimen and go into the metal again at the same end at points nearer the axis.

The Influence of an Iron Shell upon the Magnetic Behavior of a Short Cylinder within It

Some mild steel cylinders of the dimensions of the cores used in the combinations already described do not show the phenomenon of anomalous magnetization very strikingly when used by themselves, and it seemed desirable to make a series of tests upon a short piece of very soft steel about 3 centimeters in diameter, without and with shells. I have used very mild steel of various kinds for most of the observations mentioned in this paper, because it is much more homogeneous than the best procurable wrought iron, which is apt to include patches of oxide and slag which hinder the free passage of eddy currents in directions perpendicular to the grain of the material.

Tables XI, XII, and XIII, give the flux through this core when the

slowly applied current I, which created the field $H = 4\pi nI/10$ within the solenoid, was in action, after it had been gradually reduced to zero, and after it had been suddenly destroyed. The numbers in the columns headed N, N′, N″, belong respectively to the cases where the core had no shell, where the shell of soft mild steel had the diameters

TABLE XI

Current On

H	N	N′	N″
40	+ 3110	+ 747	+ 681
65	+ 5450	+1120	+1012
176	+14480	+2560	+2330
370	+31200	+5360	+4740
452	....	+6720	+5850

TABLE XII

Slow Break

H	N	N′	N″
20	+ 8	− 12	− 25
50	+ 20	− 42	− 97
100	+ 35	−142	−173
150	+ 51	−191	−204
200	+ 66	−216	−223
300	+ 92	−233	−235
400	+105	−248	−242

TABLE XIII

Quick Break

H	N	N′	N″
40	−18	+ 73	+ 74
65	−23	+106	+138
176	−18	+115	+209
378	+11	+ 30	+220
576	+42	+ 14	+216

4.45 and 3.30; and where the shell made of fine, soft, varnished iron wires had diameters of 4.90 and 3.35 centimeters respectively.

In the solid shell the alternating eddy currents which Wiedemann had in mind may encircle the core, but this would not be possible in the core made of shellacked wire. It is evident that both shells exert a strong demagnetizing effect upon the core, even while the current is running in the solenoid. All the results here tabulated agree in general with those quoted in the last section. In the case of the solid shell, the moment of core and shell taken together will usually be reversed after certain strengths of current, but this is never the case

when the shell is finely divided. Figure 18 shows a typical instance. The solid core is surrounded by a wire shell, and when the exciting current is suddenly destroyed the core has a reversed magnetization for values of H up to about 350 gausses, as is shown in curve OQR. The whole flux through the combination of core and shell as indicated

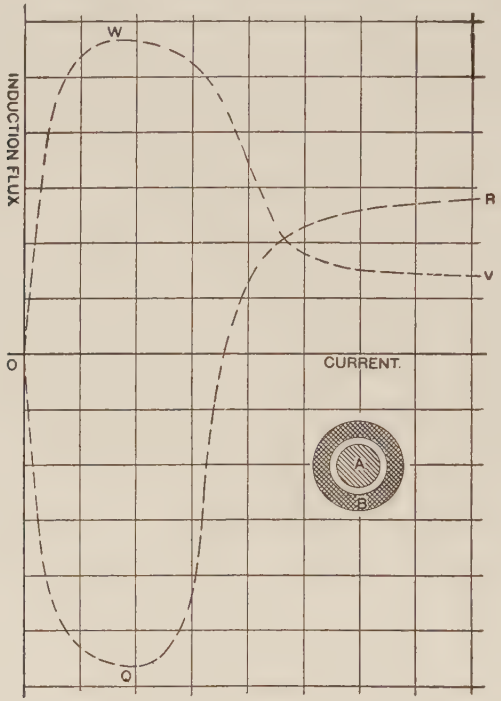


FIGURE 18. — A represents a solid core 12 centimeters long, and B, a shell made of fine, soft, varnished iron wire. This combination was magnetized in a long solenoid by a current gradually applied, and then this current was suddenly interrupted. OWV represents the induction flux through the central section of shell and core, OQR the flux through the core alone. The vertical unit is 50 maxwells, the horizontal unit 100 gausses.

by the curve OWV is never negative, though for high excitations, when the flux through the core is strongly positive, the flux through the shell itself may be small. In one case under an exceedingly high excitation, the line corresponding to OQR bent down again something like the curves of Figure 13, but the observations were so difficult to

manage that I did not attempt to follow this out in other cases. A value of H above 1700 or 1800 is hard to maintain without heating the metal and the solenoid employed unduly, which masks the effect to be studied.

The Influence of a Thick Copper Shell upon the Magnetic Behavior of a Short Cylinder within It

Many years ago Fromme enclosed a stout piece of soft iron which he was testing in a thin shell or shield of copper and was able to prove that this shell did not prevent the iron from showing anomalous magnetization when the magnetizing field about it was suddenly destroyed. Our experiences agree with his if the shell is very thin, but seem to show that a thick enough copper shell will always prevent a reversal of the magnetization in a soft iron core inside. The records of experiments on two or three specimens of soft steel with shells of different thicknesses will make clear the nature of the phenomena.

Table XIV gives some results obtained in using a mild steel cylinder 1.9 centimeters in diameter and 12 centimeters long, with a copper shell of the same length, and with diameters of 3.80 and 2.90 centimeters. The second and third columns give the fluxes in maxwells through the central cross section of the iron when the exciting current has been slowly, and quickly, brought to zero.

TABLE XIV

Core inside Shell

H	Slow Break	Quick Break
22	+14.9	+ 5.5
39	+29.6	+13.3
60	+57.2	+24.3
170	+73.3	+45.0
255	+79.0	+52.7
430	+91.7	+72.5

When the shell was removed the fluxes were those given in Table XV.

TABLE XV

Core alone

H	Slow Break	Quick Break
22	+13.5	− 9.6
39	+25.3	−17.7
100	+54.0	−14.2
218	+64.0	+13.6
430	+73.8	+48.3

The eddy currents in the copper made the gradual reduction of the current by the introduction of resistance into the circuit more continuous and prevented the magnetizing field from vanishing suddenly when the circuit was broken.

A freshly annealed piece of Bessemer steel 1.6 centimeters in diameter and 12 centimeters long was tested alone, and inside each of two

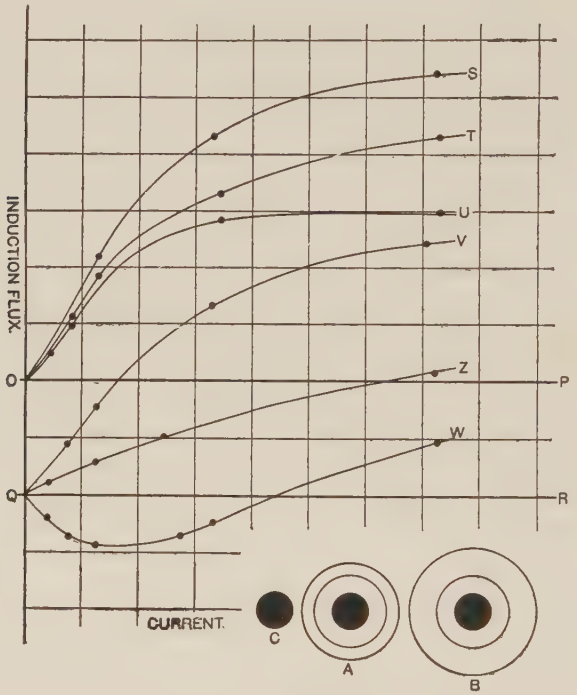


FIGURE 19. — A soft core was used successively with a thick copper shell, with a thinner shell, and without any shell. OS, OT, OU show the remanent fluxes when the magnetizing fields were slowly removed; QV, QZ, QW the fluxes if the field had been suddenly destroyed.

copper shells of its own length, with wall thicknesses of 1.20 centimeters and 0.47 centimeter respectively. Table XVI shows the fluxes through the central cross section of the iron for slow removals of the excitation. The fluxes for the thick shell, the thinner shell, and for the core without any shell, are given in the columns headed A, B, C.

Table XVII gives the corresponding figures for the case where the exciting current was suddenly destroyed.

TABLE XVI

Slow Break

H	A	B	C
38	+ 45.5	+ 42.4	+ 37.6
64	+ 84.2	+ 71.5	+ 67.0
168	+162.4	+126.0	+108.0
360	+207.0	+163.0	+112.5

TABLE XVII

Quick Break

H	A	B	C
22	+ 17.3	+ 7.0	−15.5
39	+ 33.4	+13.8	−27.4
64	+ 60.8	+20.6	−34.3
168	+127.0	+37.0	−18.9
360	+168.0	+81.5	+35.5

Only a few typical numbers are here given but these and many others are used in plotting the curves given in Figure 19. The line QW which has QR as its horizontal axis shows the reversed magnetization when the core has no shell. OS, OT, OU show the fluxes for slow breaks, QV, QZ, QW for the quick breaks.

As has been already explained, some of our slow breaks were made continuously with the help of a specially constructed rheostat, and others, sufficiently well for the purposes, by introducing, by relatively small steps, a series of resistances into the circuit. This last process which was not, of course, perfectly continuous, was used in the experiment recorded in Tables XVI and XVII and the effect of the copper in preventing any sudden change in the induction flux in the iron is evident from the figures given.

Although many complications may occur if a piece of iron or steel to be tested has a magnetic bias, or has not been uniformly tempered, the experiments described in this paper seem to lend support to the theory that the residual moment of an originally neutral bar which has been magnetized in a solenoid, is always normal unless the current on its way to extinction oscillates to and fro. When the exciting current is

destroyed without oscillating in direction, even though the process be finished in a small fraction of a second, the remanent magnetization has the same sign as the magnetizing field. It appears that a bundle of very fine soft iron wire cannot be made to show anomalous magnetism and that a thick copper shell placed over a solid bar of magnetizable metal prevents reversals of magnetism under circumstances which would produce them if the shell were away.

It seems probable that in a short, stout rod of iron or steel exposed to a magnetizing field, the intensity of magnetization in the inner portions is less than in the outer filaments and that usually when the field is removed the direction of the polarization at the axis is opposite to that of the polarization at the outer surface. The direction of the lines at the outer surface may be normal or anomalous according to the manner in which the exciting current comes to its end, but in any case many of the lines of magnetization form closed curves wholly within the metal.

The placing of a thick iron shell either solid or constructed of fine insulated wire, about a core exposed to a magnetizing field, reduces the flux through the core, and, if the exciting current be reduced gradually to zero, the shell usually reverses the sign of the moment which the core would otherwise have had. If the circuit of the exciting current be suddenly broken, the residual magnetism of the core is often changed in sign by the presence of the shell. A finely divided iron shell never acquires anomalous magnetization when its exciting current is suddenly destroyed, but such a shell acts magnetically upon either a solid or a divided core and often reverses the sign which the core would have without it.

It is difficult to make even short, stout pieces of glass-hard tool steel show anomalous magnetization, and it is impossible to reverse the magnetism of very long pieces of soft iron where the end effects are not sensible.

The experiments of Mr. L. A. Babbitt, as well as previous experiments of our own, seem to show conclusively that none of the von Waltenhofen effects are to be looked for in massive transformer cores if these are made of fine varnished wire. I have never seen anomalous magnetism in a uniformly annealed closed ring.

It is evident that if the solenoid current in a test for anomalous residual magnetism be suddenly broken, the change in the electromagnetic field in the iron is much more rapid when the core is made of lengths of fine, varnished wire than when it is solid and eddy currents in it shield the inner filaments. Indeed, if the core be made of wires of a uniform size, the average rate of change of H with the time is roughly proportional to the area of one wire. If, however, the circuit be suddenly closed, the change in the field in the iron caused by the exciting current cannot be made instantaneous even if eddy currents be wholly shut out, and the effect of dividing the core is not so striking. If the magnetized particles of a piece of iron are imbedded in a quasi viscous medium, the rapidity of the changes in the forces acting upon the molecules should affect the magnetic properties of the iron.

My thanks are due to the Trustees of the Bache Fund of the National Academy of Sciences, who have lent me some of the apparatus used in making the observations mentioned in this paper.

XXI

THE MAXIMUM VALUE OF THE MAGNETIZATION IN IRON [1]

The first experiments on the magnetic behavior of soft iron under high excitations were made, more than sixty years ago, upon comparatively short, stout rods, so that the results were affected by the demagnetizing action of the ends of the specimen, but, even under these circumstances, several different observers [2] were able to show that if the magnetizing force to which a piece of iron is exposed be made stronger and stronger, the intensity of the resulting magnetization of the metal usually approaches a definite limit, and that this limit is practically reached in fields of such strength as are frequently used in the laboratory.

The work of Stoletow and Rowland in the early seventies of the last century, upon iron rings or toroids, made the true meanings of H, B, and I in the iron clearer, and since that time many persons [3] have attempted to determine the limiting value (I_∞), of I, as H is made to increase indefinitely. I_∞ is now sometimes called the *specific magnetism* of the material.

From some of his early work, to which he applied a peculiar method of extrapolation, Rowland inferred that in the case of soft iron, I_∞ must be about 1390, whereas Fromme obtained the value 1510 in 1873, and Stefan, 1400, in the following year. In 1881, however, Fromme got the value 1737 for one specimen, and in 1884, Weber,

[1] *Proceedings of the American Academy of Arts and Sciences*, vol. xlix, no. 2, June, 1913.

[2] Joule, *Phil. Mag.*, vol. ii, 1839; Mueller, *Pogg. Ann.*, vol. lxxix, 1850; vol. lxxxii, 1851; Koosen, *Pogg. Ann.*, vol. lxxxv, 1852; Dub, *Pogg. Ann.*, vol. xc, 1853; G. Wiedemann, *Pogg. Ann.*, vol. c, 1851; vol. cvi, 1859; vol. cxvii, 1862.

[3] Rowland, *Phil. Mag.*, vol. xlvi, 1873; vol. xlviii, 1874; Stefan, *Wiener Berichte*, 1874; vol. xcvii, 1888. *Wied. Ann.*, vol. xxxviii, 1889; Fromme, *Wied. Ann.*, vol. xiii, 1881; Ewing and Low, *B. A. A. S. Report*, 1887; *Phil. Trans.*, vol. clxxx, 1889; H. E. J. G. Du Bois, *Phil. Mag.*, vol. xxix, 1890; Roessler, *Elektrotechnische Zeitschrift*, vol. xiv, 1893; Jones, *Wied. Ann.*, vol. liv, 1895; vol. lvii, 1896; Gumlich, *Elektrotechnische Zeitschrift*, vol. xxx, 1909; Peirce, *Proceedings of the American Academy*, vol. xliv, 1908; *Am. Journal of Science*, vol. xxviii, 1909; Weiss, *Journal de Physique*, May, 1910; Hadfield and Hopkinson, *Jour. Inst. Elect. Eng.*, vol. xlvi, 1911.

exposing a long rod in a solenoid to a field which had an intensity of only 900 gausses before the iron was introduced, found the corresponding value of I to be 1700.

In 1887, Messrs. Ewing and Low introduced a new and most ingenious method for experimenting upon slender isthmuses of iron and steel under very high excitations and showed that different specimens of soft iron often behaved very differently in very strong fields. For one brand of fine Swedish iron, they found the final value of I to be only 1620, while for a certain kind of Bessemer steel, the value I_∞ was as high as 1770.

Du Bois published in 1890 the results of a series of experiments upon iron in very intense fields the strengths of which he had determined by optical means. In order to obtain, for each brand of material, the Kerr's constant which he needed, he first examined an ellipsoidal test piece of the material in much weaker fields, in a solenoid. In a typical case, the soft iron ellipsoid of revolution was 18 cm. long and 6 mm. in diameter at the center. The solenoid was 30 cm. long and consisted of 1080 turns of insulated wire wound in twelve layers of about 4 cm. mean radius. The field intensity at a point 9 cm. from the center of the solenoid was about 6 per cent less than at the center

TABLE I

H'	I	H'	I
100	1410	600	1680
200	1522	800	1695
300	1590	1000	1703
400	1630	1300	1712
500	1661		

and this introduced a correction into the formula for I. The moment acquired by the bar when it was under excitation was determined, after the effect of the current in the solenoid had been compensated for, by the indications of a magnetometer in Gauss's A Position with respect to the specimen. The correction for the ends of the ellipsoid was made by the use of the formula $H' = H - 0.052\ I$. We shall find it convenient to refer to these results later on in this paper and some of them appear in Table I.

In the stronger fields the results were not so regular, for the specimen was magnetized between the poles of a powerful electromagnet

and the fields were far from uniform. The final value of I which Du Bois obtained lay somewhere between 1700 and 1750.

A paper by Roessler in the *Elektrotechnische Zeitschrift* for 1893 describes some experiments very like those made by Du Bois with the solenoid mentioned above. Roessler's solenoid was 1 meter long and consisted of 16 layers of wire 3 mm. in diameter. The mean radius of the solenoid was about 5.5 cm. and the field at a point on the axis 25 cm. from the centre was about 1 per cent less than at the centre itself. The test piece was an ellipsoid 50 cm. long and 1 cm. in diameter. The results which Roessler obtained for a certain specimen of so-called "soft iron" are given in Table II.

TABLE II

H'	I	H'	I
414	1645	848	1679
481	1663	919	1681
516	1670	991	1681
587	1675	1062	1685
741	1679	1276	1683
777	1681	1312	1687

The values of I_∞ published in 1896 by E. T. Jones, who magnetized a short, slender wire of the material to be tested between the conical pole pieces of an electromagnet of the Du Bois form, ranged as high as 1818; and the results of the joint work of Du Bois and Jones, printed in 1899, gave values of I_∞ between 1780 and 1850.

Weiss, in 1907 and 1909, experimented upon small ellipsoids of revolution made of iron, nickel, and cobalt, placed symmetrically between the flat pole pieces of a powerful electromagnet. Each ellipsoid was about 9 mm. long and 3.5 mm. in diameter. The gap between the pole pieces was about 6 cm. long and the diameter of the magnet core was 15 cm. An excitation of 94000 ampere turns corresponded to a field of about 9000 gausses in the gap centre. The small ellipsoid was suddenly drawn out of the field through a hole in the axis of one of the pole pieces and the flux change in a test solenoid outside the iron was determined. Weiss's values of I_∞ were 1731 and 1706.

Gumlich in 1909 made a series of extremely accurate determinations of the final value of I in soft iron by the Isthmus Method, using

an electromagnet of the Du Bois form, which was furnished with two soft pole pieces fastened together with the isthmus between them and capable of being rotated together about a horizontal axis perpendicular to the pole axis, so as to reverse suddenly the sign of the magnetization in the test piece. Each specimen was 28 mm. long and 3 mm. in diameter. To make sure that the lines of induction in the test piece were throughout parallel to each other, Gumlich sometimes used soft iron rings slipped over the specimen. Gumlich's value of I_∞ was 1725.

In December, 1910, Messrs. Hadfield and Hopkinson printed the results of a very carefully carried out and very elaborate investigation into the question whether, in such combinations of iron and less magnetic substances as are in practical use, the specific magnetism of any piece of the material multiplied by the mass of the piece is simply equal to the sum of the products obtained by multiplying the mass of each constituent in the specimen by its specific magnetism. They came to the conclusion that, although this rule seems not to hold in certain alloys of iron, nickel, and manganese, it is really fulfilled in many practical cases.

They used a modification of the Isthmus Method very skilfully, employing an electromagnet like, if not identical with, the magnet which Ewing and Low had, and which was made for the first isthmus experiments, under the direction of W. Low, Esquire, of Balmakewan.

Hadfield and Hopkinson had at command a large number of alloys specially made at the Hecla Works, for research purposes, and the analyses of their test pieces are therefore beyond question. They found that in their annealed iron-carbon steel, where other elements were nearly absent, the specific magnetism was less than for their standard iron by a percentage equal to about six times the percentage of carbon. In such a case they assumed that there are two constituents, pure iron, and iron carbide (Fe_3C) in mechanical mixture, the percentage of the carbide present being 15.5 times that of the carbon in the steel. The "pure iron" used as a standard was a sample of Swedish iron (Maker's mark "S. C. I.") containing less than 0.2 per cent of impurities. Of this they used two specimens: one was 6.26 mm. long, and 3.18 mm. in diameter and weighed 0.385 grams; the other was 15.92 mm. long and 3.19 mm. in diameter, and its weight

was 0.99 grams. Both yielded the same value (1680) for the specific magnetism. Table III, obtained from measurements of one of the curves given by Hadfield and Hopkinson, reproduces their results sufficiently well. Most of their pieces of steel were slightly less dense than the S. C. I. iron and their final values of I_∞ give the magnetization vector per unit volume of matter of the same density as the iron.

It is now possible to get in the market large pieces of iron which has less than 0.03 % of impurities all told, and I have used Norway iron

TABLE III

HADFIELD AND HOPKINSON'S VALUES OF THE SPECIFIC MAGNETISM OF IRON AND CARBON ALLOYS

Percentage of Carbon	Specific Magnetism	Percentage of Carbon	Specific Magnetism
*	1680	2.0	1480
0.5	1630	2.5	1430
1.0	1580	3.0	1380
1.5	1530	3.5	1330

TABLE IV

SATURATION VALUES OF THE MAGNETIZATION VECTOR IN IRON

Rowland	(1873–78)		1390
Stefan	(1874)		1400
Fromme	(1873)		1510
Fromme	(1881)		1737
Weber	(1884)	(H = 900)	1700
Ewing and Low	(1887–89)		1620 to 1740
Du Bois	(1890)	(H = 1200)	1710
Roessler	(1893)	(H = 1276)	1688
Jones	(1896)		1818
Du Bois and Jones	(1899)		1780 to 1850
Weiss	(1907)		1731
Gumlich	(1909)		1725
Peirce	(1909)		1738 to 1751
Weiss	(1910)		1706
Hadfield and Hopkinson	(1911)		1680

99.87 % pure, as well as many other specimens of nearly this excellence. In some cases the specific magnetism seemed to be about 1740 and in others less than 1700. For these determinations I have usually employed some form of Isthmus Method.

If the lines of magnetic induction in a slender homogeneous cylinder, made of perfectly soft iron, are known to be straight and parallel to the generating lines of the cylinder, we may infer that the induction

vector, — which in this case must be solenoidal and lamellar in the metal, — has the same intensity throughout the space considered. If, moreover, the lines of force in the air about the cylinder and near it on all sides seem to be straight, we may believe, since the tangential components of the magnetic force and the normal component of the induction are continuous at the surface of the iron, that the lines of force and induction in the metal are straight and parallel to the lines of the cylinder and to the lines just outside the metal in the air. If, therefore, by means of a test coil of very fine insulated wire wound tightly around the cylinder, and a somewhat larger coaxial coil which does not extend into any portion of the air where the lines of force are not straight, we determine B in the metal and H in the closely surrounding space, the ratio of the two may be supposed to give the value of the permeability in the iron. This is the theory that underlies one form of the Isthmus Method of measuring the value of the magnetization vector in the metal at high excitations. If the results are to be satisfactory, great care must be taken to make sure that the magnetic lines just outside the isthmus are really straight in the region to be used, and the dimensions of the test coils must be determined with the aid of trustworthy comparators with great accuracy. The larger test coil must be mounted upon some sort of support, if it is to keep its form unchanged, and the choice of material for a spool is very narrow. No brass or copper that I have ever tried is unmagnetic in very strong fields; paraffine wax and ebonite are often paramagnetic and introduce errors into the readings. Silk insulation for the wire of which the test coils are made is inadmissible without careful examination and even shellack, when dried from an alcoholic solution, is almost always strongly magnetic.

The form of bobbin used by Ewing and Low requires a fresh outer test coil for each specimen, but the little rods inserted at the ends into holes in the pole pieces, as in the work of Gumlich, or the shorter rods butted against the faces of the pole pieces, as in the work of Hadfield and Hopkinson, do not have this disadvantage.

If the lines of force in the air about the isthmus are practically straight for one excitation they often cease to be so when the intensity of the field is much increased. If, with soft iron pole pieces the lines are parallel for a soft iron bobbin, they may not be even approxi-

mately so for a bobbin of fairly hard steel, as I have found to my cost in a somewhat large experience.

These and other difficulties lie in the way of anyone who attempts to use the Isthmus Method in its original form, and every modification of it, whatever advantages it has, usually introduces some new problems. Notwithstanding all this, the method is a most useful one and has a much wider application than has usually been given it. No other way that has been proposed of making magnetic measurements at very high excitations is nearly so good, and the test pieces now employed are small and convenient to make.

There is still in many cases some uncertainty in the determination of H, and Hadfield and Hopkinson discuss the subject in trying to account for the differences between their results [1] and those of Gumlich, obtained at the Reichsanstalt. Moreover, there is sometimes irregularity in the values of I measured by the Isthmus Method [2] for a single specimen. For these reasons, it has seemed to me worth while to push the use of the solenoid for magnetizing test pieces farther than has yet been done, to make sure that there are specimens of metal in which I is higher than 1700 even in much weaker fields than those which the Isthmus Method furnishes. This is especially desirable if we wish to be able to determine the constitution of a large mass of steel by a quick measurement of the specific magnetism of a small test piece in an electromagnet arranged for the purpose, as has been proposed.

According to the molecular theory of magnetization of Weber and Ewing, the molecules, which lie with their magnetic axes in all directions when the metal is in the neutral state, tend to turn in the direction of any magnetic field to which the iron may be exposed, though they are hindered from doing so by the interaction of the molecules themselves. When, however, the applied field is made strong enough to overcome these intermolecular forces, in large measure, all the axes of the elementary magnets point practically in the same direction. It is evident, therefore, that unless the applied field affects the moments of the elementary magnets of which the metal is made up, the magnetic moment (I) of the metal per unit volume should remain

[1] *Jour. Inst. Elect. Eng.*, Dec., 1910, p. 253.

[2] Ewing's *Magnetic Induction in Iron and Other Metals*, Tables XI, XII.

nearly constant after the excitation has gone beyond a certain large value. This maximum magnetization is very different in different metals and we may well consider it as characteristic of a material.

Apparatus and Method of Procedure

Figure 1 shows diagrammatically the general arrangement of the great array of apparatus used in making the measurements described in this paper. This apparatus was adjusted and some of it constructed by Mr. John Coulson of the scientific staff of the Jefferson Laboratory, who has worked with me at every stage of the investigation, and to whose skill and patience I am deeply indebted. Many details are omitted from the figure. The devices for demagnetizing the specimens to be tested will be described later on.

It is evident that in such measurements of magnetic flux changes as are necessary in the work described in this paper, it is of fundamental importance that the ballistic galvanometers used be correctly calibrated, and we used a number of standards of mutual inductance, most of them rather larger than those commonly employed for such purposes, since our rather slowly moving galvanometer was not very sensitive. We had in all seventeen mutual inductances for our calibrations. Of these, five have been measured for us this year at the United States Bureau of Standards, and seven others are of such forms that their values may be calculated by well-known methods. We found one of Dr. Campbell's Variable Standards of Mutual Inductance (which was very kindly lent to us by Professor Kennelly), most useful. It proved to be very accurately calibrated, and it agreed closely at all points with the standards determined for us at Washington.

An elaborate series of comparisons of our inductances occupied Mr. Coulson and myself for more than two months, because we found that three or four of those which, according to our computations based upon their geometrical forms, should have certain values, seemed to have slightly different values, though they did not seem to be quite constant. This phenomenon puzzled us at first and gave us much trouble, but we believe, after all our work, that the ebonite used as a core in three of them is very slightly susceptible in a strong magnetic field, that the split thick brass tube used as a core for one of our sole-

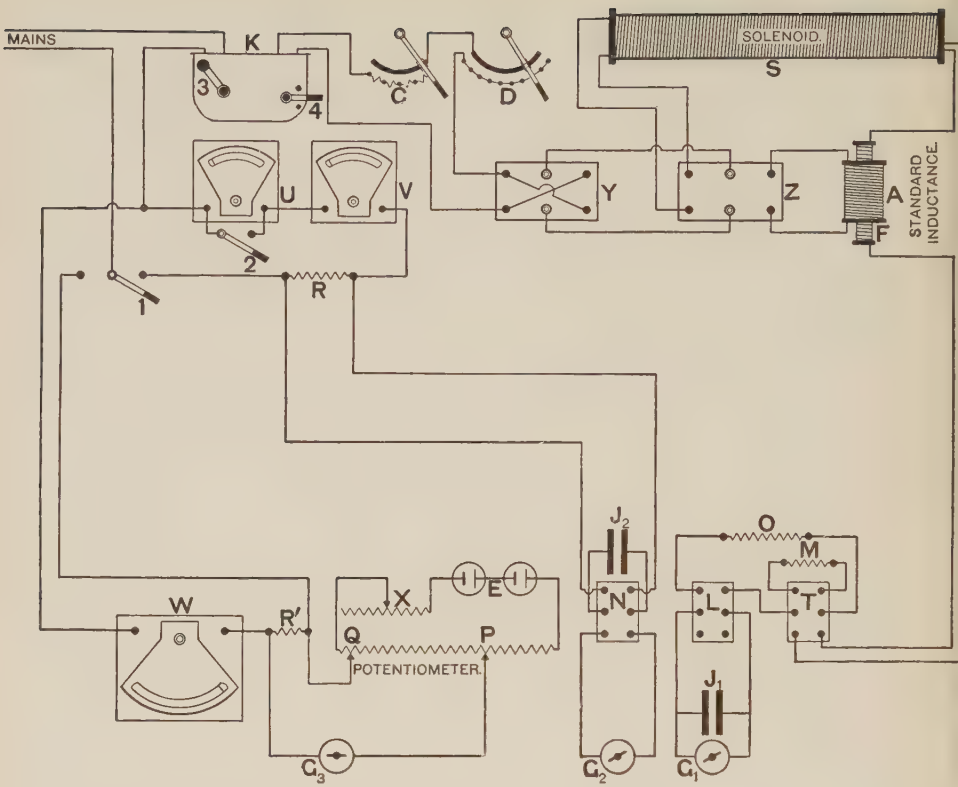


FIGURE 1. — This figure shows diagrammatically the general arrangement of some of the apparatus used in making the observations described in this paper. The elaborate devices for demagnetizing the specimens are omitted for simplicity.

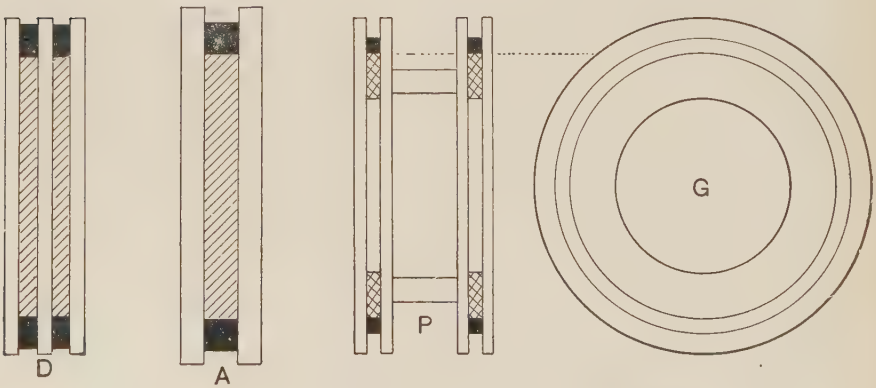


FIGURE 2. — Three standards of mutual inductance

noids is sufficiently paramagnetic to affect the field inside it perceptibly, though another solenoid constructed in a similar manner seems free from this difficulty, and finally, that the white silk triple covering of some of our wire is hygroscopic and that when a closely wound coil of it is damp, there may be a very little leakage from turn to turn through the insulation, under very strong excitation. In any event, we have eliminated all error from these sources, and we believe that the inductances of the standards we have finally used may be depended upon to at least the twentieth of one per cent.

The shapes of three of our standards are shown in Figure 2. In D, the three larger plates (and the shaded cores) are of plate glass about 29 centimeters in diameter. The cores were mounted by Mr. Thompson in an engine lathe, and were ground for about two days, under a constant flow of soda water, by a rapidly turning carborundum wheel fastened to the tool post and driven by its own motor, while the lathe moved slowly. In this way the plates were made very accurately circular. *A* is also wound upon a plate glass spool, but the two coils are wound together from two spools of wire triply covered with white silk. *P* consists of two spools with plate glass ends, but the shaded cores are ebonite rings. *G* shows a side view of *P*.

The magnetizing solenoid consists of about 300 kilograms of triply covered Number 10 copper wire wound uniformly with great care, by Mr. George W. Thompson, upon a massive brass spool 186.2 centimeters long in inside measurement. The inner coil has 8117 turns in 14 layers, and a resistance at room temperatures of about 7.7 ohms. The outer coil, of slightly different wire, has 5872 turns in 10 layers, and a resistance of about 9.8 ohms. The field intensity at different points of the axis was found for a given current in each layer separately, and it appeared from combining the results that a current of one ampere sent through the whole inner coil gives rise to a field of intensity 54.71(3) gausses at the center and 54.60(5) gausses at a point 50 centimeters from the center, on the axis. A current of one ampere sent through both coils in series creates an electromagnetic field of intensity 94.19(5) gausses at the center and 93.77(5) gausses at a point 50 centimeters from the center, a difference of nearly one half of one per cent. The outside diameter of the solenoid is a little less than 20 centimeters.

For currents up to 31.5 amperes, corresponding to a field of about 2900 gausses, the coils were used in series attached to the 550 volt circuit of the Harvard University plant. For stronger fields of 5000 gausses or more, the coils could be attached in parallel to this circuit with a standard amperemeter in each branch. For the preliminary experiments, fields stronger than 4600 gausses were not needed.

The thick, solid-drawn brass tube upon which the wire was wound carried a stream of tap water to keep the specimen at a constant temperature. The test coil was wound upon the test piece after the latter had received a very thin film of varnish. The test coil, after it had been made, was varnished and the whole was then placed for about half an hour in a stream of hot air to harden the coating. The leads were enclosed in a very thin tube of rubber, the test coil was covered with a rubber shield, and melted paraffine wax was then run into the ends of this shield so as to keep the test coil absolutely dry. In this manner all leakage from turn to turn of the triply silk-covered wire of which the test coil was made was avoided. In many cases two test coils were wound side by side upon each specimen, but the results, after we had learned how to make the coils properly, were so nearly identical for both coils that we sometimes used only one. In all cases the differences, if there ever were any real differences, were far smaller than the unavoidable errors of ballistic galvanometer reading.

The reversal of a strong current in the circuit of a solenoid with so great an inductance as this one, has to be managed carefully. The main reversing switch, when it was slightly pulled, automatically put the solenoid in parallel with a non-inductively wound resistance higher than its own, and, after the handle was raised higher, broke the main circuit so that the discharge from the solenoid could pass through the auxiliary resistance. The process was inverted when the switch handle was pushed down. This switch (Figure 3, Plate 8) was designed and made by Mr. Coulson.

To prove that the field in the solenoid, when a given current passes through the circuit, is really what it should be, according to the calculation, a very carefully made test coil without iron was placed in series with the secondary of a standard of mutual inductance, and the field was thus measured. By this means it was shown that there was no appreciable leakage between the turns of the solenoid — a very

common fault of the exciting coils of electromagnets — and that there was not enough iron in the brass of the reel to affect the field strength sensibly. So far as we can determine the fact by our many and repeated tests, the solenoid has not been injured by use and is very perfect. It is firmly mounted on a solid oak frame so that its axis is horizontal and perpendicular to the meridian.

The dimensions of the iron test pieces and of the standard inductances were obtained with the help of a set of micrometer screw gauges by Brown and Sharpe. The smallest one of these was used for determining the diameters of the specimens and of the coils wound upon them. The accuracy of this gauge was tested by a comparator (which had a screw by Gaertner), and by another comparator (by Zeiss) which reads directly to microns.

An illustration, the case of a specimen of American Ingot Iron, will show how much error would be introduced into the value of the specific magnetism of the iron by a given error in measurement of its dimensions. The length was 100 cm., the diameter of the bare iron, 1.278(5) cm., and the mean diameter of the test coil, 1.326 cm. The coil consisted of 100 turns of copper wire triply covered with white silk, and as the dimensions show, the flux through it was $128.32\ B + 9.8\ H$. The last term which shows the correction for the air flux linked with the coil is relatively small at feeble excitations, and even when H rises to 2800 and B becomes about 24,500, the whole term is less than 1 % of the flux through the iron. Moreover, the value of the term may be found to within one twenty-fifth of its value without trouble. An error of 0.001 in measuring the diameter of the iron might make an error of three units in the last place in the value of the specific magnetism and this makes it desirable to use exactly round rods. The piece here described was cut out of a large bar with great skill, at the works of Messrs. Barbour and Stockwell.

At high excitations, the corrections for the effect of the ends of the cylindrical test pieces are, of course, much less than those which according to theory and to the formulas of Du Bois and of Shuddemagen are necessary in low fields. I shall hope to discuss this matter at length in another paper, and need only state here that the correction for a piece of the dimensions used was practically negligible in fields of strength above 2000 gausses.

The rods to be annealed were first packed tightly in fine iron filings in a piece of pipe the ends of which were closed by screw caps, and the whole was carefully supported perpendicular to the meridian in a special gas heater where it would be exposed to several hundred flames driven by a power compressor. In this manner a piece 150 cm. long could be heated very uniformly. After the specimen had been kept for perhaps an hour at a temperature considerably above the critical point of the iron it could be then allowed to cool very slowly *in situ*, protected from magnetic action.

If a slender rod of iron be placed inside a long solenoid which is in the secondary circuit of a powerful open-core transformer, and if, while the primary circuit is attached to the alternate current mains, the secondary coil of the transformer be slowly drawn off the core and the primary coil, by help of some mechanical device, it is possible to send through the solenoid a long series of currents alternating in direction and gradually decreasing in intensity, and thus to demagnetize the iron rod very well. We had an apparatus of this kind permanently connected with our apparatus, but it was not shown in Figure 1 lest the diagram be too complex.

When the direction of a strong electrical current in the circuit of the large solenoid (S) in which the iron rods to be tested were magnetized, was suddenly reversed, some time was needed to establish the new current in its full value, and the change in the magnetic flux through the test coils wound upon the rods was not complete until after several seconds. This fact, due to the large inductance in the circuit, made it unsafe to employ a ballistic galvanometer of ordinary type for measuring this flux change, and we had recourse to a long-period instrument of a kind which has been used for a number of years in the Jefferson Laboratory. The particular galvanometer (G) we chose had a period of 156 seconds, which was quite long enough for our purposes, but we had a much more slowly moving instrument at hand in case of need. Any fairly long throw of G could be determined with an error of less than one tenth of one per cent, and we could do better than this by careful repetition. G is shown in Figure 4, Plate 8.

The main currents in the solenoid circuit were measured with the help of a series of Weston amperemeters (two of which are shown diagrammatically as U and V in Figure 1) properly arranged for the

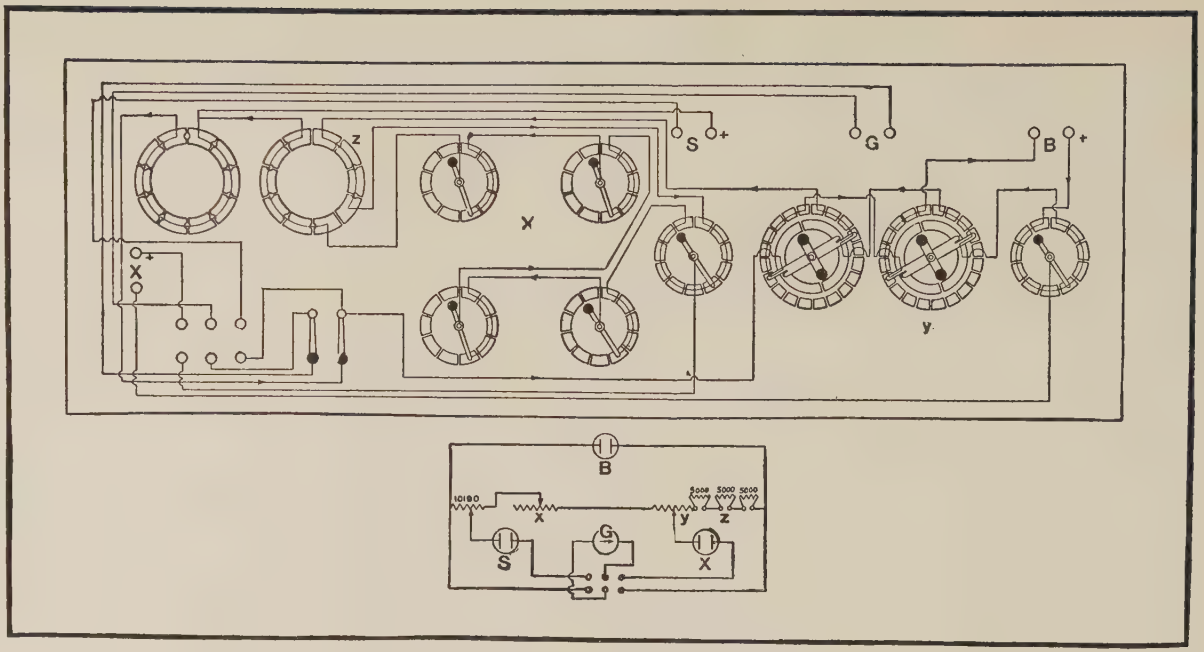


FIGURE 5. — The standard potentiometer

special intensity ranges we needed, but the accurate determination of large currents was made by aid of a potentiometer (Figure 5) with standard cadmium cells, which measured the potential drop across a standard one hundredth of an ohm resistance (R) by Crompton, which had been tested against another standard by Wolff. The largest

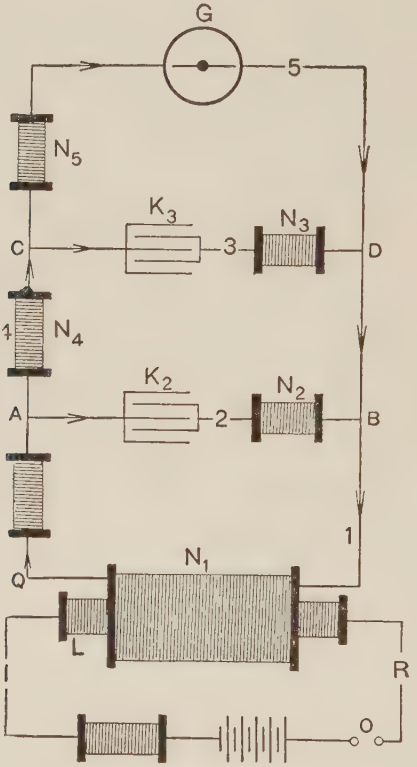


FIGURE 6

currents we used could not very well be allowed to run very long through the coils, because the amount of heat set free in the circuit was enormous. Indeed, with an energy expenditure of more than fifty kilowatts, the heating problem, in spite of running water in the core of the solenoid, needed careful consideration. As a matter of fact, the only difficulty we finally encountered was a slight falling off of our largest currents with repeated throws, owing to a little increase

in the resistance of the circuit, and this came at a place where the flow of inductance through the test coil changed very slowly with H. To save time we arranged a standard condenser (Elliot Brothers, No. 72) so that it became automatically charged at the terminals of R just as the main switch was reversed, and the charge could then be measured at our ease four or five seconds after the switch had been thrown over. By these means we avoided the delay which would have resulted if we had been obliged to read the amperemeters before the reversal.

Besides this slowly moving ballistic galvanometer, we used three other mirror galvanometers, one for the condenser throws, one for the potentiometer, and one for the accurate comparison of our inductances, and in addition, a large standard laboratory amperemeter (W), by Weston, which could be checked at any instant against the potentiometer. This beautiful instrument has an engine divided scale 31 cms. long.

At very high excitations, the reversal of any switch of ordinary construction gives rise to a very unpleasant explosion, and we often made use of a large controller (K) constructed by the General Electric Company for use upon electric cars. This was very kindly lent to us by Mr. F. W. Lieberknecht, and served an excellent purpose. We do not need to describe a large number of auxiliary amperemeters and galvanometers used in our work.

The Use of Condensers in the Inductive Secondary Circuit

The inductance in the secondary circuit, which contained the test coil or coils, the secondary coils of the inductance standards, and the coil of the large ballistic galvanometer, was usually considerable and the strain upon the insulation of the wire was sometimes large when a powerful current in the primary circuit was suddenly reversed. Occasionally, there seemed to be some leakage in this circuit, so we introduced a number of condensers into the circuit in the attempt to reduce the stress. What the exact effect of such condensers in a complex circuit will be when the breaking arc in the primary circuit is oscillatory, it is usually impossible to predict, because some necessary data are wanting or because the literal equations are of too high a degree

to be solved, but certain general facts are clear. The following analysis treats some questions, as applied to a circuit taken for illustration, which are really, perhaps, too elementary to need any discussion.

Figure 6 represents two neighboring circuits: —

(*a*) A primary circuit of total resistance R, and total self inductance L, which contains a constant battery of voltage, V, and carries a

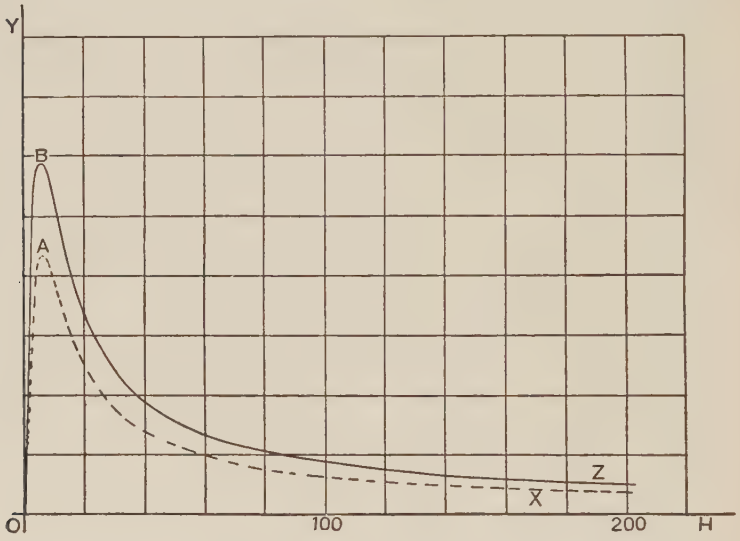


FIGURE 7. — This figure shows the forms of the curves obtained by plotting the permeability and the susceptibility of a certain kind of soft iron against the exciting field. These curves, OBZ, OAX, are drawn to different scales.

current I. This circuit is furnished with a gap (O) which may be closed or opened at pleasure.

(*b*) A secondary circuit of several branches, which has no battery, but which is linked with the primary circuit by the mutual inductance, M. The branch AQB or (1), which is directly coupled with the primary, has a resistance R_1, a total self inductance N_1, and carries a current I_1. The branches AB, CD, or (2), (3), contain condensers of capacities K_2 and K_3 respectively. Their resistances are R_2 and R_3, their self inductances, N_2 and N_3, and they carry currents I_2 and I_3. The branches (4) and (5) have no condensers. Their resistances are

R_4 and R_5, their self inductances, N_4, N_5, and their currents, I_4 and I_5. The current in DB, which has a negligible resistance, is, of course, I_4.

If accents are used to denote differentiations with respect to the time, an easy application of Kirchhoff's Laws to these two circuits leads to the equations:

$$
\begin{aligned}
&V - L\cdot I' - M\cdot I_1' = R\cdot I,\\
&-M\cdot I' - N_1\cdot I_1' - N_4\cdot I_4' - N_5\cdot I_5'\\
&\qquad = R_1\cdot I_1 + R_4\cdot I_4 + R_5\cdot I_5,\\
&-M\cdot I' - N_1(I_2' + I_3' + I_5') - N_2\cdot I_2' - Q_2/K_2\\
&\qquad = R_1(I_2 + I_3 + I_5) + R_2\cdot I_2,\\
&-M\cdot I' - N_1(I_2' + I_3' + I_5') - N_4(I_3' + I_5') - N_3\cdot I_3' - Q_3/K_3\\
&\qquad = R_1(I_2 + I_3 + I_5) + R_4(I_3 + I_5) + R_3\cdot I_3,\\
&I_1 = I_2 + I_4 = I_2 + I_3 + I_5,
\end{aligned}
\tag{1}
$$

or

$$
\begin{aligned}
&(L\cdot I' + R\cdot I) + M\cdot I_2' + M\cdot I_3' + M\cdot I_5' = V,\\
&M\cdot I' + (N_1\cdot I_2' + R_1\cdot I_2) + (N_1\cdot I_3' + N_4\cdot I_3' + R_1\cdot I_3 + R_4\cdot I_3)\\
&\quad + (N_1\cdot I_5' + N_4\cdot I_5' + N_5\cdot I_5' + R_1\cdot I_5 + R_4\cdot I_5 + R_5\cdot I_5) = 0,\\
&M\cdot I'' + (N_1\cdot I_2'' + N_2\cdot I_2'' + R_1\cdot I_2' + R_2\cdot I_2' + I_2/K_2)\\
&\quad + (N_1\cdot I_3'' + R_1\cdot I_3') + (N_1\cdot I_5'' + R_1\cdot I_5') = 0,\\
&M\cdot I'' + (N_1\cdot I_2'' + R_1\cdot I_2') + (N_1\cdot I_3'' + N_4\cdot I_3'' + N_3\cdot I_3''\\
&\quad + R_1\cdot I_3' + R_4\cdot I_3' + R_3\cdot I_3' + I_3/K_3)\\
&\quad + (N_1\cdot I_5'' + N_4\cdot I_5'' + R_1\cdot I_5' + R_4\cdot I_5') = 0.
\end{aligned}
\tag{2}
$$

If, for I we write $I_0 + V/R$, the second number of the first equation becomes zero, while all the equations remain otherwise unchanged in form, and it follows that every one of the currents satisfies a single linear differential equation of the sixth order with constant coefficients and that if a, b, c, d, e, and h are the roots of the equation formed by equating to zero the determinant,

$$\begin{vmatrix} Lx+r & Mx & Mx & Mx \\ Mx & N_1x+R_1 & \left\{\begin{matrix} N_1x+N_4x+R_1 \\ +R_4 \end{matrix}\right\} & \left\{\begin{matrix} N_1x+N_4x+N_5x \\ +R_1+R_4+R_5 \end{matrix}\right\} \\ Mx^2 & \left\{\begin{matrix} N_1x^2+N_2x^2+R_1x \\ +R_2x+1/K_2 \end{matrix}\right\} & N_1x^2+R_1x & N_1x^2+R_1 \\ Mx^2 & N_1x^2+R_1x & \left\{\begin{matrix} N_1x^2+N_4x^2+N_3x^2 \\ +R_1x+R_4x+R_3x \\ +1/K_3 \end{matrix}\right\} & \left\{\begin{matrix} N_1x^2+N_4x^2+R_1x \\ +R_4x \end{matrix}\right\} \end{vmatrix} \tag{3}$$

then

$$\begin{aligned} I &= Ae^{at} + Be^{bt} + Ce^{ct} + De^{dt} + Ee^{et} + He^{ht} + \frac{V}{R} \\ I_2 &= \alpha_2 Ae^{at} + \beta_2 Be^{bt} + \gamma_2 Ce^{ct} + \delta_2 De^{dt} + \epsilon_2 Ee^{et} + \eta_2 He^{ht} \\ I_3 &= \alpha_3 Ae^{at} + \beta_3 Be^{bt} + \gamma_3 Ce^{ct} + \delta_3 De^{dt} + \epsilon_3 Ee^{et} + \eta_3 He^{ht} \\ I_5 &= \alpha_5 Ae^{at} + \beta_5 Be^{bt} + \gamma_5 Ce^{ct} + \delta_5 De^{dt} + \epsilon_5 Ee^{et} + \eta_5 He^{ht}. \end{aligned} \tag{4}$$

If these values be substituted in one of the Kirchhoff equations above and the coefficients of the different exponential expressions separately equated to zero, it will appear that the α's, β's, γ's, δ's, ϵ's, and η's are determinate functions of the constants of the circuit and in no way dependent upon the manner in which the currents are managed.

The other six constants (A, B, C, D, E, H) have to be computed from a knowledge of the electrical conditions which determine any problem concerning these two fixed circuits.

If, for instance, there is no current in any branch of the circuits at the outset, and if the gap, O, be suddenly closed at the origin of time, the values of the constants for all positive time satisfy the equations:

$$\begin{aligned} A + B + C + D + E + H &= -\frac{V}{R} \\ \alpha_2 A + \beta_2 B + \gamma_2 C + \delta_2 D + \epsilon_2 E + \eta_2 H &= 0 \\ \alpha_3 A + \beta_3 B + \gamma_3 C + \delta_3 D + \epsilon_3 E + \eta_3 H &= 0 \\ \alpha_5 A + \beta_5 B + \gamma_5 C + \delta_5 D + \epsilon_5 E + \eta_5 H &= 0, \end{aligned} \tag{5}$$

and, after these have been solved, it is easy to compute the whole flow of electricity through the galvanometer, for

$$\int_0^\infty I_5\, dt = -\left(\frac{\alpha_5 A}{a} + \frac{\beta_5 B}{b} + \frac{\gamma_5 C}{c} + \frac{\delta_5 D}{d} + \frac{\epsilon_5 E}{e} + \frac{\eta_5 H}{h}\right). \tag{6}$$

If, however, when the primary current has its steady value, V/R, and there are no currents in the secondary circuit, the primary resistance be instantaneously changed from R to R', at the time $t = 0$, there is no sudden change in the current in any branch, but for all subsequent time the constants are determined by the equations:

$$\begin{aligned} A + \quad B + \quad C + \quad D + \quad E + \quad H &= \frac{V}{R} - \frac{V}{R'} \\ \alpha_2 A + \beta_2 B + \gamma_2 C + \delta_2 D + \epsilon_2 E + \eta_2 H &= 0 \\ \alpha_3 A + \beta_3 B + \gamma_3 C + \delta_3 D + \epsilon_3 E + \eta_3 H &= 0 \\ \alpha_5 A + \beta_5 B + \gamma_5 C + \delta_5 D + \epsilon_5 E + \eta_5 H &= 0,^1 \end{aligned} \tag{7}$$

and it is evident that every one of the quantities A, B, C, D, E, H, I given by (7) has a value which bears to the corresponding value given by (5) the ratio $(R - R')/R'$, and the same relation holds between the whole discharges through the galvanometer in the two cases. If the gap be instantly opened so that R' is infinite, when the current in the primary circuit is V/R and there are no secondary currents, the galvanometer throw is equal, but opposite in sign, to the throw caused by suddenly closing the gap when all the currents are 0.

The electrokinetic energy for the coupled circuits is

$$\begin{aligned} T = {} & \tfrac{1}{2} L \cdot I^2 + M \cdot I \, (I_2 + I_3 + I_5) + \tfrac{1}{2} N_1 \, (I_2 + I_3 + I_5)^2 \\ & + \tfrac{1}{2} N_2 \cdot I_2^2 + \tfrac{1}{2} N_3 \cdot I_3^2 + \tfrac{1}{2} N_4 (I_3 + I_5)^2 + \tfrac{1}{2} N_5 \cdot I_5, \end{aligned} \tag{8}$$

so that the electrokinetic momenta are

$$\begin{aligned} p &= L \cdot I + M \, (I_2 + I_3 + I_5) \\ p_2 &= M \cdot I + N_1 \, (I_2 + I_3 + I_5) + N_2 \cdot I_2 \\ p_3 &= M \cdot I + N_1 \, (I_2 + I_3 + I_5) + N_3 \cdot I_3 + N_4 \, (I_3 + I_5) \\ p_5 &= M \cdot I + N_1 \, (I_2 + I_3 + I_5) + N_4 \, (I_3 + I_5) + N_5 I_5. \end{aligned} \tag{9}$$

If, when I has the value $I = V/R$, and there are no other currents, the gap be instantly opened, I suddenly drops to 0, and I_2, I_3, I_5, which were 0, suddenly acquire initial values which may be determined by

[1] The quantities a, α, b, β, c, γ, etc., have different values in (5) and in (7), so that A, as determined from (7) and the last two equations of (1), is $(R - R')/R'$ times the same function of the new constants that the A of (5) is of the old ones. The relation is, in so far, merely a formal one. A simple integration of the second equation of the set (1), with respect to the time, gives directly, however, the numerical values of the total flow of electricity through the galvanometer in the cases to which (5) and (7) correspond.

the fact that the electrokinetic momenta, p_2, p_3, p_5, which before the change were equal to $P_0 = MV/R$, are not altered by the impulse. After the gap is opened, the currents in the branches obey the system of equations (2), but the initial values of these currents are to be found from the equations:

$$\begin{aligned} (N_1 + N_2)\, I_2 + N_1 I_3 + N_1 I_5 &= P_0 \\ N_1 I_2 + (N_1 + N_3 + N_4)\, I_3 + (N_1 + N_4)\, I_5 &= P_0 \\ N_1 I_2 + (N_1 + N_4)\, I_3 + (N_1 + N_4 + N_5)\, I_5 &= P_0. \end{aligned} \tag{10}$$

If, then, Δ denote the determinant of the coefficients,

$$\begin{aligned} \Delta = {} & N_1N_2N_3 + N_1N_3N_4 + N_1N_3N_5 + N_1N_2N_5 + N_1N_4N_5 \\ & + N_2N_3N_5 + N_2N_3N_4 + N_2N_4N_5, \end{aligned}$$

and the values of I_2, I_3, and I_5, just after the gap is opened, are

$$(N_3N_4 + N_3N_5 + N_4N_5)P_0/\Delta,\ N_2N_5P_0/\Delta, \text{ and } N_2N_3P_0/\Delta. \tag{11}$$

The total amount of electricity carried by the currents I_2, I_3, and I_5 are 0, 0, and Ω; and to find Ω we may integrate the second equation of the system (2) with respect to the time from 0 to ∞ and use the initial values of I_2, I_3, I_5 just found. This procedure leads to the equation:

$$\Omega = \int_0^\infty I_5 \cdot dt = \frac{P_0}{R_1 + R_4 + R_5}, \tag{12}$$

and this is evidently the same result that would have been obtained for the whole discharge through the galvanometer, if the branches (2) and (3), with their condensers, were removed from the secondary circuit. It is easy to compute the sudden loss of energy when the gap is opened.

Results

For the purposes of the investigation here described, we used about twenty-five different brands of iron obtained from several different sources. Of these, five gave values of I larger than 1700 for comparatively low excitations of about 2800 gausses.

About ten of our specimens were described by the dealers as "Bessemer" and showed similar micrographs. Most of these were in no way remarkable. For excitations of about 2700 they gave values

of I of about 1675 in the average and might be expected to give 1685 for fields of strength 5000. One specimen (No. 10) was quite different from the others. For $H = 2730$ the corresponding value of I was 1727. This result is based upon several different determinations made upon different days, and during the interval the rod was once annealed. A long series of annealings, however, reduced the permeability so that finally the I corresponding to $H = 2600$ fell to about 1700. One specimen of wrought iron which we have annealed a great number of times shows no permanent change in permeability although at one stage this fell by more than one per cent temporarily, and was restored by the next annealing.

According to my experience during the last few years with a good many pieces of so-called "Norway Iron," about one specimen in three of those bought without care in the open market may be expected to have a specific magnetism considerably above 1700. Different portions of the same large bar may have very different permeabilities, however, as one may readily believe after an examination of a series of micrographs which always show a considerable amount of slag. I believe that an occasional small piece such as would be used for an isthmus might be found to have a specific magnetism three or four per cent above the best value to be found in a bar. I have myself encountered two isthmuses which gave 1790 and 1751 respectively, in spite of my best efforts to reduce what seemed to me at the time impossibly large values. Some small specimens used by other observers have shown even greater values than this. In the case of a rod a meter long and twelve mm. in diameter, however, I have never found an average value much above 1740.

Much of the wrought iron to be had in the market, though very useful to blacksmiths, contains such an amount of slag that the continuity of the metal is seriously affected and the permeability of the mass is not very high. Such are the specimens of "Farnley Iron," marked here "F," the "Taylor Iron" and the "Best Refined Iron," which show low values of I in moderate fields. It is possible to get in the open market "Norway Iron" of great purity. One specimen which I used showed, upon analysis, no nickel, cobalt, manganese, or tungsten. It contained less than 0.03 % of carbon, less than 0.047 % of phosphorus, less than 0.03 % of silicon, and less than 0.003 % of

sulphur. This, however, does not compare in purity with the "American Ingot Iron," which contains less than 0.03 % of impurities all told, and shows a very remarkable micrograph.

Our specimens of this iron were very kindly furnished by Dr. Percy W. Bridgman, who has been using this material in some of his experiments upon the behavior of metals under very high pressures.

TABLE V

VALUES OF THE MAGNETIZATION VECTOR IN VARIOUS KINDS OF IRON

No.	Material	Diameter	*H*	*I*
1	"Taylor Iron"	0.975	2490	1645
2	"Taylor Iron"	0.998	2685	1654
3	Bessemer	1.266	2695	1663
4	Bessemer	1.269	2675	1687
5	Bessemer	1.269	2790	1677
6	Bessemer	0.950	2880	1671
7	Bessemer	0.950	2825	1666
8	Bessemer	1.269	2370	1685
9	Bessemer	0.634	2800	1673
10	Bessemer	0.632	2730	1727
11	"American Ingot Iron"	1.277	2760	1708
11	"American Ingot Iron"	1.277	4395	1711
12	"American Ingot Iron"	1.279	2780	1725
12	"American Ingot Iron"	1.279	4545	1735
13	Norway	1.078	2865	1686
14	Norway	0.969	2885	1685
15	Norway	1.302	2770	1661
16	Norway	1.280	2770	1727
16	Norway	1.280	4360	1742
17	Norway	0.294	2970	1735
18	"Cold Rolled Shafting"	1.269	2680	1693
19	"Cold Rolled Shafting"	1.269	2740	1678
20	"Best Refined Iron"	1.280	2740	1620
21	"Best Refined Iron"	0.999	2825	1589
22	"Best Refined Iron"	1.002	2690	1611
23	Drill Rod	0.794	2790	1533
24	"R"		2810	1637
25	"F"	0.968	2665	1627

Plate 10 shows micrographs of two pieces. The first was in the normal state; the second had been exposed by Dr. Bridgman to a hydrostatic pressure of 17000 atmospheres for about 16 hours! The magnification is 120 diameters.

Mr. Herbert M. Boylston, of Messrs. Sauveur & Boylston, who has most kindly examined, under the miscroscope, the polished and etched

specimens of these irons, reports that Nos. 1, 2, 11, 12, 14, 20, 21, 22, and 25 contain little, if any, carbon. Of the Bessemers, with which "R" must be reckoned, No. 6 contains only about 0.05 % of carbon, while No. 10, which has a high specific magnetism, has 0.15 % and each of the other pieces about 0.10 %. The Drill Rod has about 1.10 % of carbon; the specimen, which is of very fine grain, has no slag and shows simply sorbite, pearlite, and some cementite in fine net work.

Nos. 1, 2, 20, 21, 22, 24, and 25 contain considerable slag, and this is in comparatively large masses in portions of the "Refined Iron." The relatively small amount of slag in the Norway Irons is in fine particles distributed through the mass. Nos. 11 and 12 seem to be simply ferrite.

Table VI gives some corresponding values of H and I for the specimen of "American Ingot Iron," known as No. 12.

TABLE VI

H	I	H/I
40.5	1343	0.030
68.8	1398	0.049
129.9	1476	0.088
229.8	1570	0.146
271	1597	0.170
399	1660	0.240
457	1684	0.271
871	1715	0.507
1126	1717	0.655
1569	1719	0.913
2288	1724	1.323
2747	1727	1.589
4543	1735	2.620

It is well known that if a relatively stout rod of soft iron be exposed to a strong field in a solenoid and if the magnetizing current be very suddenly broken, the direction of the residual magnetism in the rod may be opposite in sign to what it was when the current was running. If the rod be enclosed in a thick walled copper tube within the solenoid, this reversal never takes place and the sign of the residual magnetism is always normal. The moment of the rod under the new field is often larger when the current is suddenly reversed than when it is slowly reduced to zero through a constantly growing resistance before the switch is thrown over and then gradually brought to its new

strength, or when the change is made less violent by eddy currents induced in a thick copper shell around the specimen. Though it seemed possible that the results given in this paper might be slightly affected by this so-called von Waltenhofen phenomenon, we could not discover the least difference in our results whether the rod to be used was or was not surrounded by a thick copper tube, though the tube makes the throws a trifle more regular.

It is well known, also, that under low excitations the magnetic moment acquired by a rod in a solenoid under a given final excitation may be much increased if the rod be constantly tapped while the magnetic changes are taking place. So far as we can make out, this effect is entirely lacking at very high excitations. We used a large electric tapping apparatus made by Mr. Coulson to give many sharp blows per second to a brass rod butted upon the specimen in the solenoid, and Table VII shows characteristic results.

TABLE VII

Current	Flux Change when the Iron was undisturbed	Flux Change when the Iron was tapped
4.85	939a	938a
8.95	974a	974a
30.2	1013a	1013a

Some years ago I encountered three specimens of very pure Norway Iron, each of which showed a very high specific magnetism, when tested by a modification of the Isthmus Method. Each piece was about 8 cms. long and 1.26 cms. in diameter. They were presumably from different sources.

TABLE VIII

Specimen	Exciting Field	I
1	2500	1733
2	2400	1738
3	2350	1751

Each of these numbers comes from a series of closely consistent values for different field strengths, and I have no reason to think that the determinations were not good, but I consider the probable error somewhat greater in all work I have done with isthmuses than with such experiments as I have made with larger specimens, with the help of a solenoid.

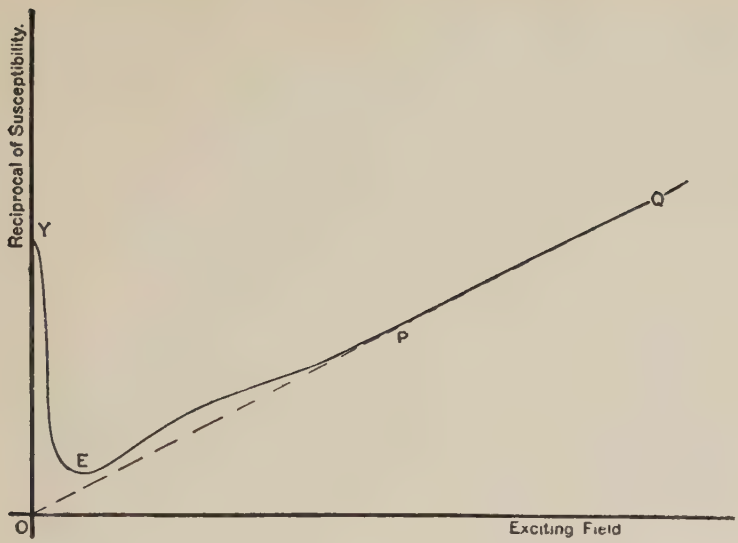


FIGURE 8 shows the form of the curve obtained by plotting the reciprocal of the susceptibility of soft iron against the intensity of the exciting field. The ratio of the abscissa of any point of the curve to the corresponding ordinate is less than the final value of I, and the tangent of the angle which the tangent to the curve makes with the ordinate axis is greater than this value except for small values of H.

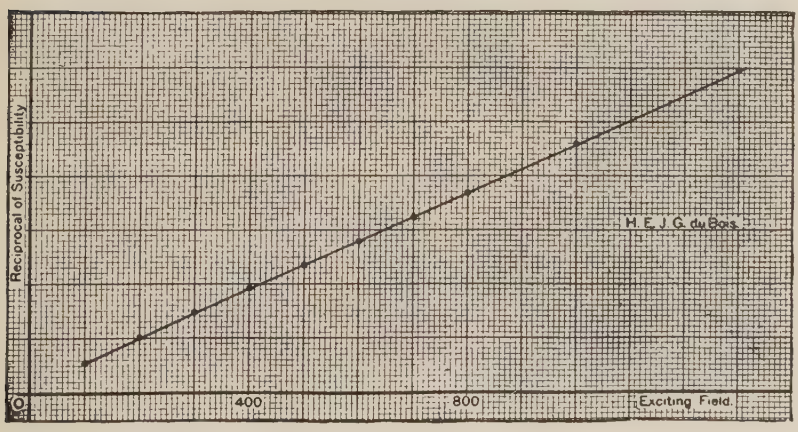


FIGURE 9.—This curve shows the results of observations made by Du Bois upon an ellipsoidal piece of soft iron 18 cms. long and 0.6 cm. in diameter at the middle.

A single slender isthmus cut from the bar from which No. 2 in this table was taken gave the very large value 1796 for I in fields above 6000, but other larger pieces from the same bar showed lower values for I. According to my experience, very small bits taken from closely adjacent regions of the same bar may have very different specific magnetisms, owing perhaps to differently arranged inclusions of slag. It is certain that pieces cut across the direction of the rolling often show different permeabilities from those of pieces cut in the

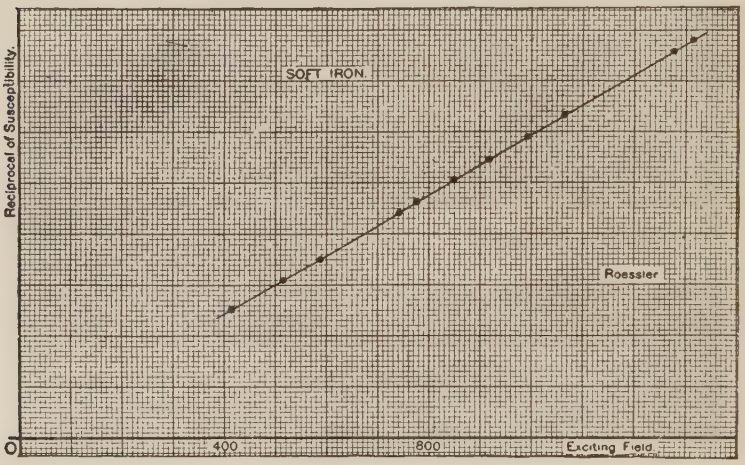


FIGURE 10. — This figure shows the results of observations made by Roessler upon an ellipsoidal piece of soft iron 50 cms. long and 1 cm. in diameter at the centre.

same region in the direction of the length of the bar. Gumlich found a piece of soft " Steirisches Eisen " about 3 cms. long and about 3 mm. in diameter which also showed the value 1796 for I.

There seems to be no doubt, therefore, that some specimens of soft iron are to be found which have materially higher maximum values of I than had the specimen used as a standard by Messrs. Hadfield and Hopkinson. Four different observers using solenoids for magnetizing their test pieces, and seven persons using other methods, have thought that they met with such pieces. This fact does not, of course, make the work of Messrs. Hadfield and Hopkinson any the less valuable, but it shows, I think, since some pieces which contain consider-

able quantities of $Fe_3 \cdot C$ have given values of I above 1720, that material bought in the open market cannot be expected to obey the law which the series of steel pieces from the Hecla Works follows.

Still, the majority, perhaps, of pieces of iron and steel bought at random will have specific magnetisms not very different from the values given as a result of experiments upon these beautiful test pieces.

If a series of observations be made by the Method of Reversals, upon a piece of iron originally in a neutral state, and if the permeabil-

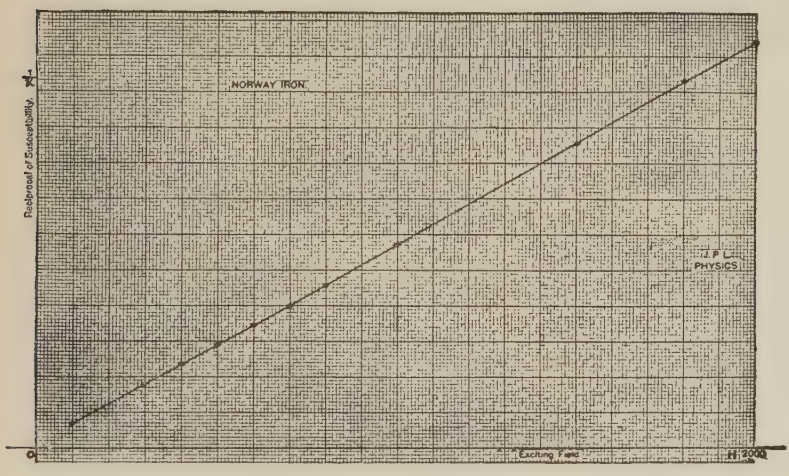


FIGURE 11 shows the results of observations made in the Jefferson Laboratory upon a rod of Norway Iron. For excitations up to about 400, the specimen was magnetized in a solenoid. For more intense fields, the determinations were made by a modification of the Isthmus Method.

ity and the susceptibility obtained in this way be plotted against the intensity of the exciting field, those portions of the resulting curves (Figure 7) which correspond to large values of H resemble hyperbolas which have the x and the y axes as asymptotes. A generation ago, therefore, it seems to have occurred to a number of persons at about the same time that if the reciprocals of the permeability and of the susceptibility were plotted against H, the curves must become finally more or less straight. It appeared upon trial that for values of H larger than 100, say, the reluctance gives a line only slightly convex

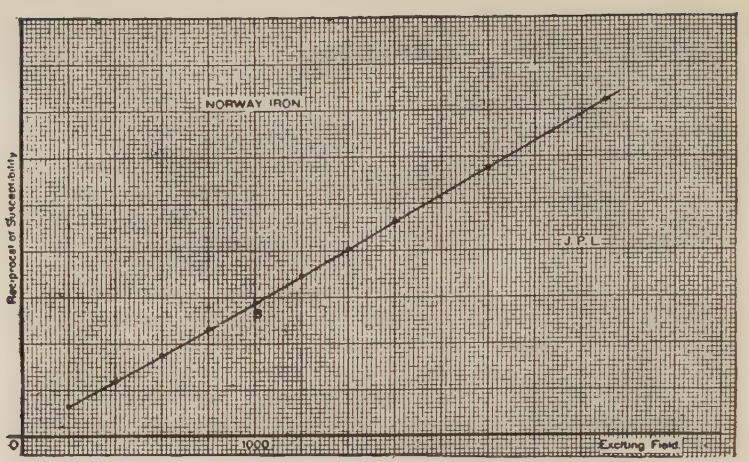


FIGURE 12 represents observations made in the Jefferson Laboratory upon a second specimen of Norway Iron.

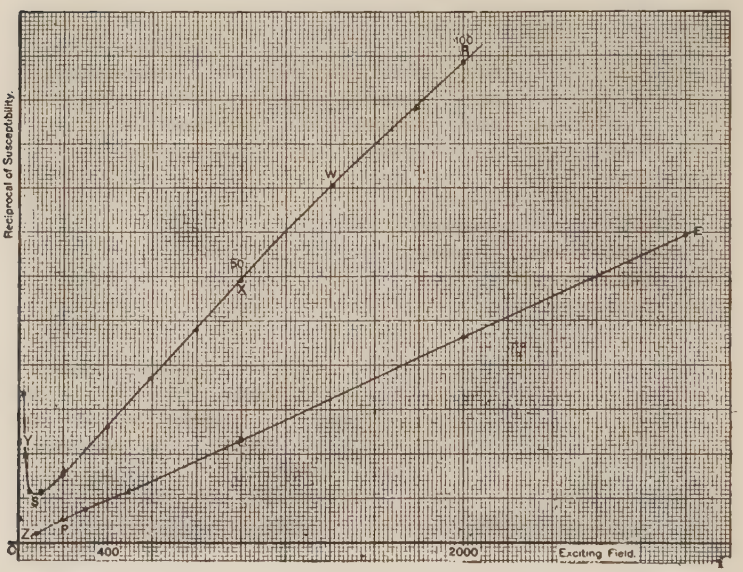


FIGURE 13. — This figure shows results obtained from tests made upon a specimen of Bessemer steel, 8.0 cms. long and 1.26 cms. in diameter. For low excitations the tests were made in a long, slender solenoid. For higher fields a modification of the Isthmus Method was used.

upwards, and that the reciprocal of the susceptibility which for comparatively weak fields has the general shape shown in Figure 8 becomes very nearly coincident with a straight line drawn through the origin under high excitation. This last function has been found useful by Professor Kennelly in his paper upon the relation between B and H in fields [1] of commercial strength. It is clear from the curve

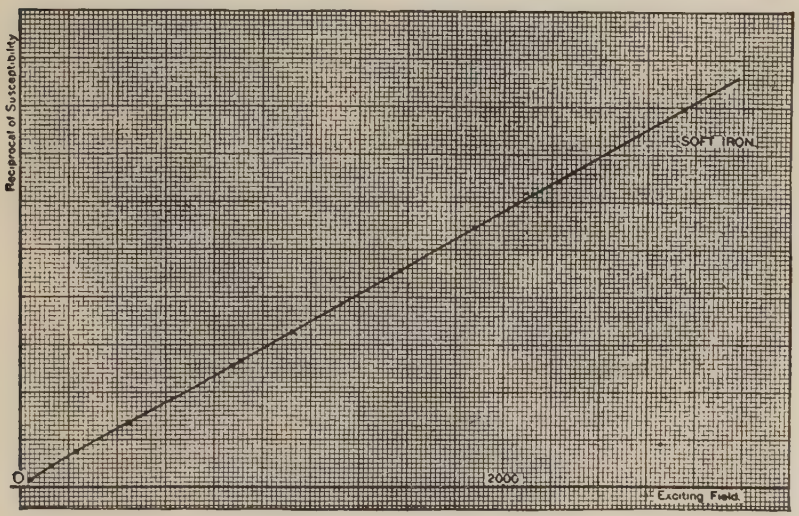


FIGURE 14. — This figure is plotted from the observations made in the Jefferson Laboratory upon a piece of "American Ingot Iron" magnetized in the solenoid. The piece was 100 cms. long and 1.279 cms. in diameter.

in Figure 8, the ordinates of which are $H/I = 1/k$, that the ratio of the abscissa of any point of the curve to its ordinate always yields a value of I somewhat less than the saturation value, whereas the slope against the ordinate axis of the tangent of the curve, after H

[1] Lenz and Jacobi, *Pogg. Ann.*, vol. xlvii, 1839; Mueller, *Pogg. Ann.*, vol. lxxix, 1850; Von Waltenhofen, *Wiener Berichte*, vol. lii, 1865; Lamont, *Handbuch d. Magnetismus*, p. 41; Sohncke, *Elektrotechnische Zeitschrift*, 1883; Ruths, *Ueber d. Magnetismus weicher Eisenxylinder*, 1876; Froelich, *Elektrotechnische Zeitschrift*, 1881, 1882, 1894; Kennelly, *Trans. Am. Inst. El. Eng.*, vol. viii; Steinmetz, *Elektrotechnische Zeitschrift*, vol. xiii, 1892; Muellendorf, *Elektrotechnische Zeitschrift*, vol. xxii, 1901; vol. xxiii, 1902; Kapp, *Electrician*, vol. xviii, 1886; Winkelmann's *Handbuch der Physik*, vol. v, p. 200; Walter, *Drude Ann.*, vol. xiv, 1904; Czermak and Hausmaninger, *Wiener Berichte*, vol. xcviii, 1889; Du Bois, *Wied. Ann.*, vol. li, 1894; Fromme, *Wied. Ann.*, vol. xiii, 1881; vol. xxxiii, 1888.

equals perhaps 200, is always greater than I. Such curves as this are especially useful when one wishes to study the saturation values of the magnetization in iron or steel.

Figures 9 and 10 show the results of plotting the reciprocals of the susceptibilities obtained by Du Bois, and Roessler in their experiments already described.

Figures 11, 12, and 13 reproduce the results of a series of measurements made two or three years ago in the Jefferson Laboratory upon specimens of Bessemer steel and of Norway Iron. For excitations up to about 400 the specimens were magnetized in a slender solenoid about five meters long, but for stronger fields a modification of the Isthmus Method was employed. Figure 14 shows some measurements made lately upon a specimen of "American Ingot Iron" magnetized in the shorter solenoid described above. Such curves become practically straight for much weaker fields in the case of some irons than in others.

I wish to express my great obligation to the Trustees of the Bache Fund of the National Academy of Sciences for the loan of some of the apparatus used in making the observations mentioned in this paper.

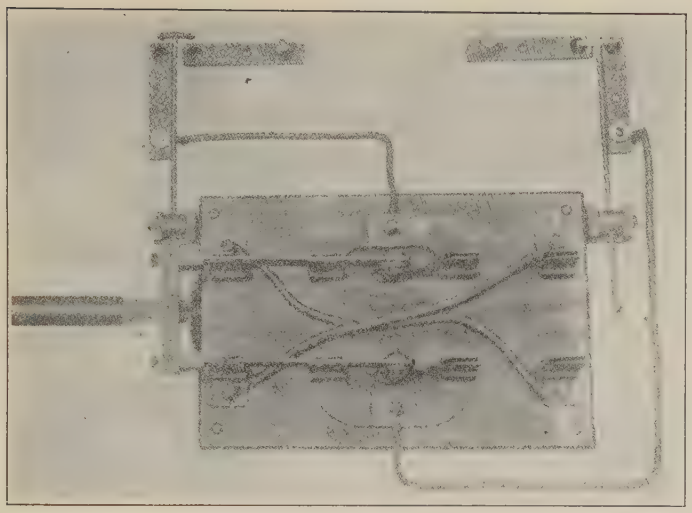


FIGURE 3. — A switch in the main circuit so arranged that if it be suddenly thrown over, the energy in the medium which accompanied the old current is spent largely in heating an auxiliary coil.

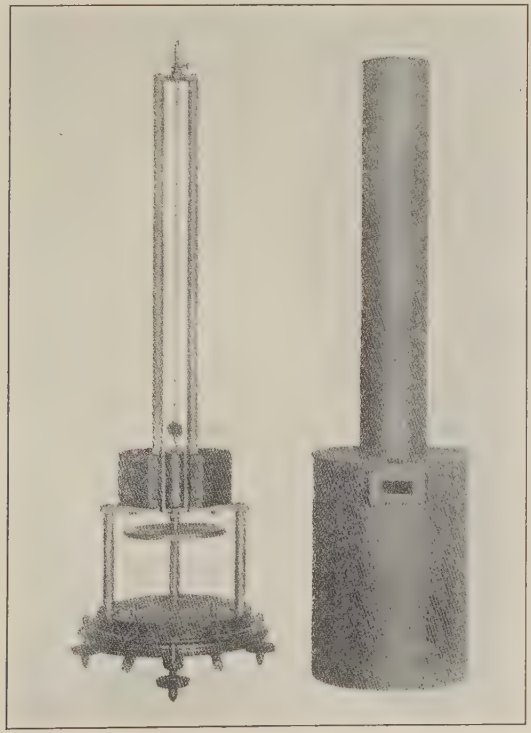
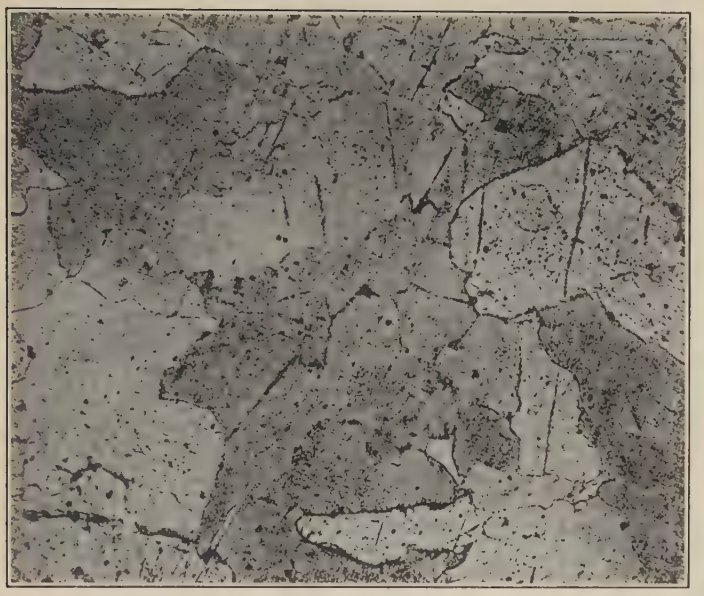
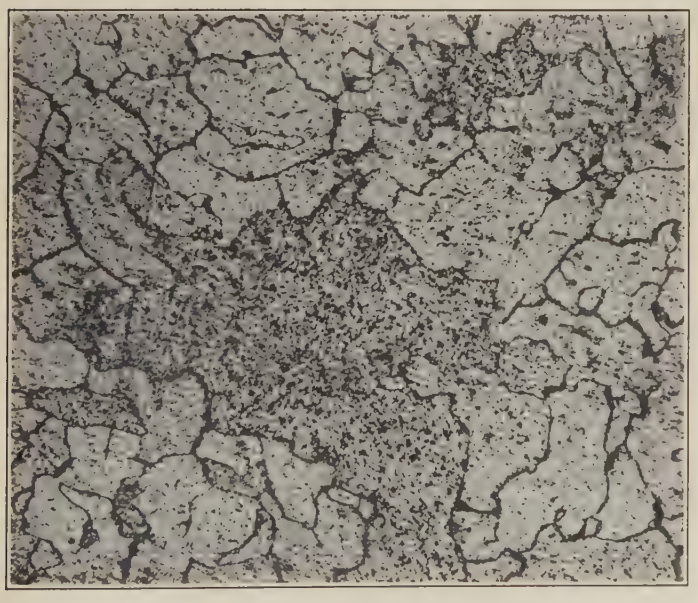
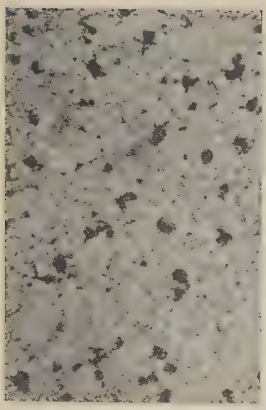


FIGURE 4. — The long-period ballistic galvanometer







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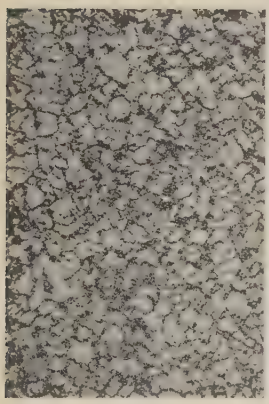

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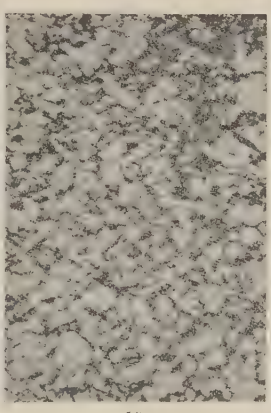

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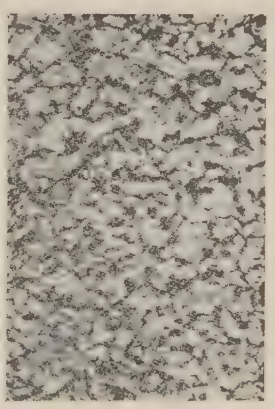

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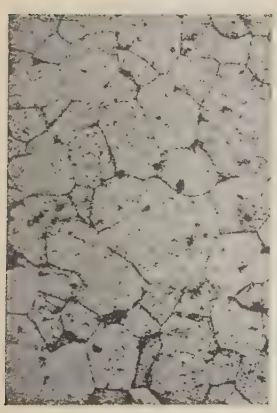

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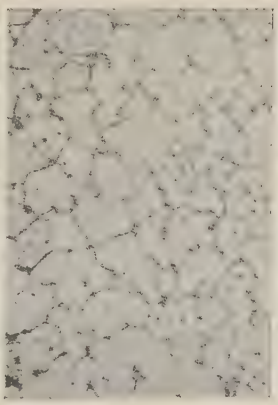

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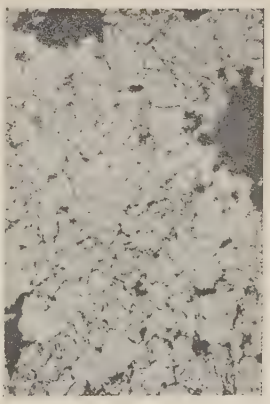

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[1] Now Professor of Physics in the University of Pittsburgh.

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